

GEOMETRIC REDUCTIVITY – A QUOTIENT SPACE APPROACH

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1. Introduction

Mumford's *Geometric Invariant Theory* or *GIT* is a major technique for finding quotients of algebraic schemes acted upon by reductive algebraic groups. It has been successful in finding solutions to moduli problems in the category of *algebraic schemes*. In the first edition (i.e., the 1965 edition) of *Geometric Invariant Theory* [M], Mumford restricted himself to algebraic schemes over fields of characteristic zero. In order to make his theory applicable over fields of arbitrary characteristic, he made the following conjecture in the Preface to the first edition of *ibid.* (a conjecture subsequently proved by Haboush [H] in 1975):

Let G be a reductive algebraic group over an algebraically closed field k . Then G is GEOMETRICALLY REDUCTIVE, i.e., for every finite-dimensional rational G -module V and a G -invariant point $v \in V$, $v \neq 0$, there is a G -invariant homogeneous polynomial F on V of positive degree such that $F(v) \neq 0$.

As a consequence, it can be shown that if $X = \text{Spec } A$ is an algebraic scheme on which a reductive algebraic group G acts, then the affine scheme $Y = \text{Spec } A^G$ is an algebraic scheme, i.e., the ring of invariants A^G is finitely generated as a k -algebra (a result of Nagata [N]) and the canonical morphism $f: X \rightarrow Y$ (induced by the injection $A^G \hookrightarrow A$) is surjective. Further, if Z is a closed G -stable subset of X , then $f(Z)$ is closed in Y , and if f separates *disjoint closed G -stable* subsets of X , i.e., given two disjoint closed G -stable subsets Z_1 and Z_2 of X , then $f(Z_1)$ and $f(Z_2)$ are also disjoint. In other words, $f: X \rightarrow Y$ is what is called a *good quotient* [S2]. These results are in fact *equivalent* to the conjecture.

A major consequence then is that Mumford's technique—as set out in [M]—for constructing the quotient (in the category of algebraic schemes) of the *semi-stable locus* of a projective algebraic scheme X on which a reductive group G acts linearly¹ works over fields of arbitrary characteristic. Recall that in the first edition of *ibid.* such quotients were only constructed when the underlying field was of characteristic zero, and over such a field reductive algebraic groups have been known, via Hermann Weyl's work (see [W]), to be linearly reductive, whence geometrically reductive. In fact in characteristic zero geometric reductivity is equivalent to the complete reducibility of finite dimensional G -modules.

Geometric reductivity (for our reductive algebraic group G) was first proved for the case of $SL(2)$ (hence $GL(2)$) in characteristic 2 by Oda [O], and in general by the second author [S1]. Haboush's proof [H] uses in an essential way the irreducibility of the Steinberg representation. There is also a different approach to the problem due to Formanek and Procesi, à priori for the full linear group [F-P], but the general case can be deduced from this (see [S5]).

¹i.e., G acts linearly on the ambient projective space in which X is embedded.

Successful as the above approach has been in solving the problem of quotients (of algebraic schemes by reductive algebraic groups), a direct attack on the quotient problem has an undoubted philosophical attraction. And success here would yield all the consequences of geometric reductivity (e.g., Nagata's result on finite generation of invariants), and almost as an after thought, also yield geometric reductivity.

Before Haboush settled the conjecture, the second author, in [S2], made an attempt to solve this conjecture following the quotient space approach and had partial success. In *ibid.* it is also shown that constructing the Mumford or GIT quotient is equivalent to constructing a quotient by a proper equivalence relation on a projective variety; in fact, proving the conjecture is equivalent to showing that a natural line bundle on this projective variety, which is known to be *nef*, is in fact *semi-ample* (i.e. a suitable power of this line bundle has no base points). Recently Sean Keel [SK] has given a very interesting criterion for a nef line bundle on a projective variety to be semi-ample in characteristic p , $p > 0$. Using this result of Keel and strengthening the methods of [S2], we give a proof of Mumford's Conjecture in this paper.

Geometric reductivity of a reductive group G is equivalent to showing that the set Y of equivalence classes of semi-stable points for a linear action of G on a projective scheme X has a canonical structure of a projective scheme (see below for the definitions and notations). Roughly speaking one can say that the proof given here (of this equivalent form of Mumford's conjecture) consists in checking the Nakai-Moishezon criterion for ampleness for a natural choice of a line bundle L on Y . A principal tool is the Hilbert-Mumford criterion i.e. a process of reduction to a maximal torus or even a 1-dimensional torus (Chapter 2, [M]), for checking stability, semi-stability etc. of points. An essential difficulty in this approach is that it is not easy to see, *à priori*, any natural scheme theoretic structure on Y . This has to be built up in stages, first as a topological space on which suitable notions of properness, morphisms etc. have to be introduced, eventually culminating in a scheme structure on Y (which is shown to be projective via L). In the case when "stable = semi-stable" this process is simpler; it is easier to show that Y is a proper scheme and the proof is, indeed, checking the Nakai-Moishezon criterion for L on Y ([S2]). In the general case one shows that there is a projective scheme which surjects onto Y ; in fact, one can find such a projective scheme Q which is "generically finite" over Y and one works with such "models" for Y (Y is the quotient of Q by a proper equivalence relation, as mentioned above). The Nakai-Moishezon criterion is to be interpreted as checking that L (i.e. the pull-back of L on Y) is "big" on Q and this is done by refining the methods of [S2]. However, this does not suffice to complete the proof of geometric reductivity for one cannot expect L to be ample on Q but only "semi-ample" and one requires some analogue

of the Nakai-Moishezon criterion for semi-ampleness. This is achieved by the work of Sean Keel ([SK]) and appealing to this work, the semi-ampleness of L on Q follows. With a little more work, the required structure of a projective scheme on Y also follows. A more comprehensive outline of proof is given in [S5].

One knows that geometric reductivity for reductive algebraic schemes holds over a general *base scheme* (see [S5] or Appendix G to Chap. 1, [M]), the proof being again based on the irreducibility of the Steinberg representation. One would also like to prove this by the quotient space approach as in this paper. For this one requires a suitable generalisation of Sean Keel's result and this seems to pose difficulties.

1.1. Conventions, Notations and Definitions. We work throughout over an algebraically closed field of *positive characteristic* $p > 0$. Thus, for example, a *variety* or a *scheme* will mean a k -variety or a k -scheme respectively. A variety means a *separated reduced finite type* scheme (over k).

We fix a semi-simple algebraic group G over k (except briefly in Section 5 where we allow G to be reductive algebraic). The aim of the paper is to show that G is geometrically reductive.

As is standard, \mathbb{G}_m denotes the multiplicative group scheme of “non-zero scalars” in k , i.e.,

$$\mathbb{G}_m := \operatorname{Spec} k[T, T^{-1}].$$

We will largely be working in a setting where G acts linearly on a projective scheme (in fact, more often than not, on a projective variety). It is convenient to have in place a terminology which will act as a shorthand for certain recurring situations.

Definitions 1.1.1. A *linear G -pair*—or simply a *G -pair*—is a pair (X, L) with X a projective scheme and L an ample line bundle on X such that G acts on X and this action lifts to a linear action on L . The G -pair (X, L) is said to be *reduced* if X is reduced. A *linear G -triple*—or simply a *G -triple*— $(X, L, \mathbb{P}(V))$ consists of a G -pair (X, L) , a finite dimensional rational G -module V , and a G -equivariant closed embedding $X \hookrightarrow \mathbb{P}(V)$ such that L is the restriction of the tautological ample bundle $\mathcal{O}(1)$ on $\mathbb{P}(V)$. The G -triple $(X, L, \mathbb{P}(V))$ is said to be *reduced* if (as before) X is reduced.

Here are some other conventions and notations not listed above:

1) We repeat that by a variety we mean a separated reduced irreducible scheme of finite type. Points on a scheme of finite type mean closed (whence k -rational) points.

2) If V is a k -vector space, then we identify V with the scheme $\operatorname{Spec} S(V^*)$, where V^* is the dual of V and $S(V^*)$ is the symmetric algebra on V^* .

3) If $(X, L, \mathbb{P}(V))$ is a G -triple, we often denote the tautological ample bundle $\mathcal{O}(1)$ on $\mathbb{P}(V)$ by the letter L . If we need to distinguish between L on $\mathbb{P}(V)$ and L on X , we use the symbols $L_{\mathbb{P}(V)}$ for the former and L_X for the latter.

4) If $(X, L, \mathbb{P}(V))$ is a G -triple, then, as is standard, \widehat{X} will denote the cone in V over X . Note that \widehat{X} is a closed G -stable subscheme of V . Note also that $\widehat{\mathbb{P}(V)} = V$.

5) The *homothety action* on \widehat{X} , for a G -triple $(X, L, \mathbb{P}(V))$ is the action of \mathbb{G}_m on \widehat{X} given by scalar multiplication.

6) A *one parameter subgroup* λ of G is a map of algebraic groups $\lambda: \mathbb{G}_m \rightarrow G$. We use the abbreviation 1-PS for “one parameter subgroup”. If G acts algebraically on a k -scheme and λ is a 1-PS of G , then we often call the resulting action of \mathbb{G}_m on X as the *action of λ on X* .

2. The Main strategy

Recall that if $(X, L, \mathbb{P}(V))$ is a G -triple, the semistable locus $X^{ss} = X^{ss}(L)$ of (X, L) is the locus of points $x \in X$ such that if \widehat{x} is a point on the cone $\widehat{X} \hookrightarrow V$ over X which represents x , then the orbit of \widehat{x} does not contain the origin $0 \in V$ in its closure. If the orbit of \widehat{x} is closed, and $\dim \widehat{x}G = \dim G$, we say x is a stable point and denote the stable locus X^s or $X^s(L)$. We refer the reader to Section 3 for a more extended discussion. In particular, there it is shown that the relation $x \sim x'$ whose graph is $\{(x, x') \in X^{ss} \times_k X^{ss} \mid \overline{xG} = \overline{x'G}\}$ is an equivalence relation, the so-called *semi-stable equivalence relation*. (See Definition 3.4.5.) Our focus, as we have pointed out earlier, is to extend the techniques of [S2] to prove that our semi-simple algebraic group G is geometrically reductive. This implies that every reductive group is geometrically reductive. In [S2, p. 550, Theorem 7.1] it is proven that if (X, L) is a G -pair with X *normal* and *projective*, whose stable locus X^s agrees with its semi-stable locus X^{ss} (both loci with respect to L), and $X = \text{Proj } R$ where $R = \bigoplus_{n \geq 0} \Gamma(X, L^n)$, then R^G is a finite type k -algebra. Moreover, if $Y = \text{Proj } R^G$, and $\pi: X \rightarrow Y$ the *rational* map induced by the inclusion $R^G \hookrightarrow R$, then the map π is regular on the semi-stable locus X^{ss} , and the resulting map $X^{ss} \rightarrow Y$ is a geometric quotient. It is also proven in [S2, p. 553, Theorem 7.2] that if $x \in X$ is a stable point (X not necessarily normal) then there exists a G -invariant non-constant homogeneous polynomial p on X such that $p(x) \neq 0$. The problem is to extend this result to semi-stable points. In this section we flesh out the main strategy and reduce the problem to finding a map of stacks $Q \rightarrow [X^{ss}/G]$ (from a normal projective variety Q) with certain properties, the most important amongst them being that the resulting map from Q to a natural stratified space Y associated with the G -action on X^{ss} is generically finite and the line bundle L_Q on Q induced by L is nef and big.

2.1. Preliminaries. We begin by giving a few definitions.

Definition 2.1.1. The G -invariant map $X^{ss} \hookrightarrow \mathbb{P}(V)$ is said to be *saturated*, if $X^{ss} \neq \emptyset$, and for semi-stably equivalent points v and v' in $\mathbb{P}(V)$, $v \in X^{ss}$ implies that $v' \in X^{ss}$. We often simply say X^{ss} is saturated when the G -invariant embedding into $\mathbb{P}(V)$ is clear.

Recall that a line bundle L on a projective algebraic scheme is *nef* if $\deg L|_C \geq 0$ for every irreducible curve on X , and it is *big* if for $n \gg 0$ the *regular* global sections of L^n define a *rational birational* map. In case X is reduced and projective, and L is nef, then it is big if and only if $L^{(r_i)}|_{X_i} > 0$ for every irreducible component X_i of X , with $r_i = \dim X_i$. (See [K2, VI.2.15 and VI.2.16].) Finally L is *semi-ample* if some positive power of it is generated by global sections. In other words, given a point $x \in X$ there is a section of a positive power of L which does not vanish at x . If L is semi-ample (say L^n is generated by global sections) and big and then the regular morphism on X induced by L^n is birational on to its image.

Definition 2.1.2. Let $(X, L, \mathbb{P}(V))$ be a G -triple. We say L is *G -semi-ample on X^{ss}* , or L is *G -semi-ample on X* , if given $x \in X^{ss}$, there is a positive integer n and an element $s \in \Gamma(X^{ss}, L^n)^G$ such that $s(x) \neq 0$. Equivalently, given $\hat{x} \in \hat{X}^{ss}$ there is a G -invariant regular non-constant homogenous function F on \hat{X}^{ss} such that $F(\hat{x}) \neq 0$.

Definition 2.1.3. We say $\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$ is a *quotient data* if $(X, L, \mathbb{P}(V))$ is a G -triple, $X^{ss} := X^{ss}(L)$, Y the *topological space* obtained by quotienting X^{ss} by the semi-stable equivalence relation, and α the resulting quotient map (cf. Definition 3.6.2 in Section 3 below). We say Δ is *saturated* if $X^{ss} \hookrightarrow \mathbb{P}(V)^{ss}$ is saturated, *reduced* if X is reduced, and *irreducible* if Y is irreducible. Δ is a *strong quotient data* (or simply a *strong data*) if it is reduced, saturated, $\mathbb{P}(V)^s \neq \emptyset$, and L restricted to the fibres of α is trivial.² (The importance of the conditions Δ saturated and $\mathbb{P}(V)^s \neq \emptyset$ is seen in Proposition 5.2.1.) We often write $(X^{ss} // G)_{\text{top}}$ for the quotient Y .

Note that if $\mathcal{Y} := (\mathbb{P}(V)^{ss} // G)_{\text{top}}$ and $\tilde{\alpha}: \mathbb{P}(V)^{ss} \rightarrow \mathcal{Y}$ is the resulting quotient map of topological spaces, then $X^{ss} \hookrightarrow \mathbb{P}(V)$ is saturated if and only if $\tilde{\alpha}^{-1}(Y) = X^{ss}$, where we regard Y as a closed subspace of \mathcal{Y} in a natural way.

Definition 2.1.4. Let $\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$ be a quotient data. A *non-empty* scheme U is said to be a *generic quotient* Δ as above if the underlying topological space of U is an open dense subset of Y such that $\alpha^{-1}(U) \xrightarrow{\text{via } \alpha} U$ is a morphism of schemes ($\alpha^{-1}(U)$ having the canonical open X^{ss} -subscheme structure

²This can be achieved by replacing L by a suitable power of L , as we show in Remark 4.1.3.

on the open set $\alpha^{-1}(U)$, and such that \mathcal{O}_U is the sheaf of G -invariant sections of the direct image of $\mathcal{O}_{\alpha^{-1}(U)}$.

We show later (see Lemma 4.1.1) that if Δ is a reduced irreducible quotient data, then a generic quotient for Δ exists. A little thought shows that by working with each irreducible component, and removing all points which lie in more than one irreducible component, a generic quotient exists for all reduced quotient data, whether irreducible or not.

Given quotient data Δ , we write X_Δ , L_Δ , $\mathbb{P}(V_\Delta)$, α_Δ , Y_Δ etc. for the various datum comprising Δ .

If Δ satisfies all the requirements of a strong quotient data except the requirement that L is trivial on the fibres of α , then, this requirement is easily achieved by replacing L by a suitable positive power of itself. Indeed, by Lemma 4.1.1 this is achieved on a dense open subset of Y , and working with the complement of U in Y , and continuing the process, by noetherian induction we achieve what we wish. One consequence is that we have a line bundle $L_{/G}$ on the algebraic stack $[X^{ss}/G]$ (see remarks in 4.2.4). We point out that $X^{ss}(L^n) = X^{ss}(L)$ (resp. $X^s(L^n) = X^s(L)$) for n positive as is readily verified among other methods by the Hilbert-Mumford criterion [S2, p. 520, Theorem 2.2] (see also [*Ibid.*, p. 519, Proposition 2.1 (2)]).

Given a reduced quotient data Δ it can be strengthened to a strong data with very little effort. Indeed replace V_Δ by $V_\Delta \oplus W$ with W a rational finite dimensional G such that $\mathbb{P}(W)^s \neq \emptyset$, and X_Δ by the closure of the inverse image of Y under the semi-stable quotient map on $(\mathbb{P}(V)^{ss} // G)_{\text{top}}$ and finally replacing L_Δ by suitable positive power of itself, we get a strong quotient data.

2.2. Zariski locally trivial principal G -bundles. The problem of showing the geometric reductivity of G is equivalent to showing that if Δ is a strong quotient data, then L_Δ is G -semi-ample on X_Δ . Indeed it is enough to show that $\mathcal{O}(1)_{\mathbb{P}(V)}$ is G -semi-ample for a finite dimensional G -module V , and we note that the quotient data $(\mathbb{P}(V), \mathcal{O}(1)_{\mathbb{P}(V)}, \mathbb{P}(V)^{ss} \xrightarrow{\tilde{\alpha}} \mathcal{Y})$ is strong. In this section we work with strong *irreducible* quotient data Δ and reduce the problem of showing L_Δ is G -semi-ample to that of finding a Zariski locally trivial principal G -bundle $P \rightarrow Q$ and a G -invariant map $P \rightarrow X^{ss}$ such that $Q \rightarrow Y$ is “generically finite” (see Definition 4.1.5), and such that if $f: Q \rightarrow [X^{ss}/G]$ is the classifying map, then $f^*L_{/G}$ is nef and big. Here $L_{/G}$ is the line bundle on $[X^{ss}/G]$ defined by $L = L_\Delta$. Other technical hypotheses are required to be satisfied by the principal bundle $P \rightarrow Q$. Here is what is needed.

Suppose $\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$ is a strong *irreducible* quotient data. We quickly summarize what we need from Section 4 so that this section can be followed. Note that Y is a stratified space. Indeed by Lemma 4.1.1 we can find a generic

quotient U for Δ , and by working with the complement $Y' = Y \setminus U$, we get a strong quotient data Δ' where $X' = X_{De'}$ is the closure in X of $\alpha^{-1}(Y')$. Therefore we can find a generic quotient U' of Δ' and U' is an open subset of Y . Proceeding in this manner we have stratified space (cf. Remark 4.1.3). The original open stratum is called the *big stratum* for this stratification. If Q is an algebraic scheme and $f: Q \rightarrow [X^{ss}/G]$ is a map of stacks, then (as Q is the base of a principal G -bundle) we have an obvious continuous map $q: Q \rightarrow Y$, and this map is a stratified map (roughly, over each stratum we have a map of schemes, with the inverse image of a non-big stratum being given the reduced structure). We can regard Y itself as a stack (see §§ 4.2, especially (4.2.1)), though it is not an algebraic stack as defined (lacking a smooth atlas) whence the above considerations on Q define a map $\gamma: [X^{ss}/G] \rightarrow Y$. Finally since Δ is a strong data, L is trivial on the fibres of α , whence its pull back to the principal bundle defined by $f: Q \rightarrow [X^{ss}/G]$ descends to Q . This means we can talk of a line bundle $L|_Q$ on $[X^{ss}/G]$ which should be regarded as the line bundle to which $L|_{X^{ss}}$ descends.

In what follows, let \tilde{X} be the normalization of X , \tilde{L} the pull back of L to \tilde{X} , and $\tilde{X}^{ss} = \tilde{X}^{ss}\tilde{L}$.

We will show in Section 7 that there exists an *irreducible normal projective variety* Q together with a map of algebraic stacks

$$(2.2.1) \quad f: Q \rightarrow [X^{ss}/G]$$

such that:

- (1) The principal G -bundle $\beta: P \rightarrow Q$ corresponding to f is Zariski locally trivial.
- (2) The map $q := \gamma \circ f: Q \rightarrow Y$ is “generically finite”. In other words, if U is a generic quotient for Δ then the map of schemes $q^{-1}U \rightarrow U$ is generically finite (see also Definition 4.1.5). By Corollary 3.6.8 it follows that $Q \rightarrow Y$ is surjective.
- (3) Set $L_Q = f^*L|_G$. Then L_Q is nef and big on Q .
- (4) If C is a closed integral curve in Q such that $q|_C$ is non-constant, then $\deg(L_Q|_C) > 0$.
- (5) Let $k(Y) := k(U)$. Then $k(Q)$ is normal over $k(Y)$ and the finite group $\Gamma = \text{Aut}_{k(Y)}(k(Q))$ acts on Q , and $q: Q \rightarrow Y$ is Γ -invariant for the trivial action of Γ on Y . There exists a generic quotient U for the data Δ such the fibres of q over U are Γ -orbits.
- (6) The action of Γ lifts to L_Q , whence a positive power of L_Q descends to $\overline{Q} := Q/\Gamma$. For definiteness, suppose r is positive and L_Q^r descends to \mathcal{L} on \overline{Q} .

Note that P is realized as the “base change” $P := Q \times_{[X^{ss}/G]} X^{ss}$, and β is the projection to the first factor. P , being smooth over the normal variety W , is itself normal. We also have a second projection which is a G -invariant map $\pi: P \rightarrow X^{ss}$. Note that $\Gamma(\overline{Q}, \mathcal{L}^n) \xrightarrow{\sim} \Gamma(Q, L_Q^{rn})^\Gamma$ for every positive integer n . In addition to the conditions above, the map (2.2.1) also satisfies:

- (7) The map $\pi: P \rightarrow X^{ss}$ takes values in an irreducible component of X^{ss} , say X_1^{ss} and the map $P \rightarrow X_1^{ss}$ is dominant. Moreover $\tilde{\pi}: P \rightarrow \tilde{X}^{ss}$ is the map induced by π , then the sections of \mathcal{L}^n , up to a suitable power of p , can be identified with the Γ -invariant sections of the pull back under $\tilde{\pi}: P \rightarrow \tilde{X}^{ss}$ of G -invariant sections of \tilde{L}^{rn} . In other words, if $t \in \Gamma(\tilde{X}^{ss}, \tilde{L}^{rn})^G$ then a suitable p -power of t , say t^n (with $n = p^m$), is in the image of the map

$$\Gamma(\tilde{X}^{ss}, \tilde{L}^{rn})^G \rightarrow \Gamma(\overline{Q}, \mathcal{L}^n)$$

which is the composite

$$\begin{aligned} \Gamma(\tilde{X}^{ss}, \tilde{L}^{rn})^G &\rightarrow [\Gamma(\tilde{X}^{ss}, \tilde{\pi}_* \tilde{\pi}^* \tilde{L}^{rn})^G]^\Gamma \longrightarrow [\Gamma(P, \tilde{\pi}^* \tilde{L}^{rn})^G]^\Gamma \\ &:= \Gamma(Q, L_Q^{rn})^\Gamma \\ &\xrightarrow{\sim} \Gamma(\overline{Q}, \mathcal{L}^n). \end{aligned}$$

2.3. Geometric Reductivity. We now show, assuming the existence of the map (2.2.1) for strong irreducible quotient data, that Geometric Reductivity of G holds. More precisely, we show that if Δ is a strong quotient data then L_Δ is G -semi-ample on X_Δ .

Fix a strong quotient data $\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$.

Lemma 2.3.1. *Suppose Δ is irreducible. Let $f: Q \rightarrow [X^{ss}/G]$ be the map (2.2.1) satisfying conditions (1)–(7) of §2.2. Assume L_Q descends to a line bundle $L_{\overline{Q}}$ on \overline{Q} . If L_Q is semi-ample on Q then*

- (1) $L_{\overline{Q}}$ is semi-ample.
- (2) There is a factorization of $q: Q \rightarrow Y$ given by the commutative diagram

$$(2.3.2) \quad \begin{array}{ccc} Q & \xrightarrow{\varphi} & \overline{Q} \\ q \downarrow & \searrow \bar{q} & \downarrow \psi \\ Y & \xleftarrow{q_W} & W \end{array}$$

where φ is the natural quotient map $Q \rightarrow Q/\Gamma$, ψ is birational and W is projective and normal, with $L_{\overline{Q}}^n$ descending to an ample line bundle L_W on W for a suitable positive integer n , and q_W has finite fibres. Moreover, if $\bar{q}: \overline{Q} \rightarrow Y$ is the composite $\bar{q} = q_W \circ \psi$, there exists a generic quotient

U of Δ such that the maps $(\bar{q})^{-1}(U) \rightarrow U$ and $q_W^{-1}(U) \rightarrow U$ are bijective continuous maps, and $(\bar{q})^{-1}(U) \rightarrow q_W^{-1}(U)$ is an isomorphism of schemes.

Proof. Since $Q \rightarrow Y$ is Γ -invariant for the trivial action of Γ on Y , we have a map $\bar{q}: \bar{Q} \rightarrow W$ such that $\bar{q} \circ \varphi = q$. According to property (5) in §§2.2, we have a generic quotient U for Δ such that $q^{-1}(U) \rightarrow U$ has fibres which are Γ -orbits. It follows that $(\bar{q})^{-1}(U) \rightarrow U$ is bijective.

Now $L_{\bar{Q}}$ is semi-ample. Moreover it is big and nef since L_Q is big and nef. By replacing $L_{\bar{Q}}$ by a positive power of itself, we may assume $L_{\bar{Q}}$ is base point free. Since $L_{\bar{Q}}$ is big, the projective map induced by it is birational on to its image, whence we have the birational map $\psi: \bar{Q} \rightarrow W$ and $L_{\bar{Q}}$ descends to an ample line bundle L_W on W . In fact

$$(2.3.3) \quad W = \text{Proj } S$$

where $S = \bigoplus_{n \geq 0} S_n$ is the graded ring given by $S_n = \Gamma(\bar{Q}, L_{\bar{Q}}^n)$. Moreover, W is normal, since \bar{Q} is. In particular, the fibres of $\psi: \bar{Q} \rightarrow W$ are connected. For $w \in W$, we claim that $\bar{q}(\psi^{-1}(w))$ is a point in Y . Suppose not. Then we have points a and b in $\psi^{-1}(w)$ such that $\bar{q}(a) \neq \bar{q}(b)$. Let C be a closed integral curve in the connected space $\psi^{-1}(w)$ which passes through a and b . Then C is generically finite on to its image C' in Y . By condition (4) satisfied by Q (see §§2.2) we see that $\deg L_{\bar{Q}}|_C > 0$. On the other hand $L_{\bar{Q}} = \psi^* L_W$, and hence $L_{\bar{Q}}|_C$ is a trivial line bundle. This gives us a contradiction. We therefore deduce a map $q_W: W \rightarrow Y$ such that the diagram above commutes. It remains to show that q_W has finite fibres. If not, we have a closed integral curve C in W which maps to a point in y in Y . Since L is trivial on $\alpha^{-1}(y)$, it follows that $L_{\bar{Q}}$ is trivial on the proper transform of C in \bar{Q} by property (7) enjoyed by the map f in (2.2.1), whence $\deg(L_W|_C) = 0$. On the other hand L_W is ample. This is a contradiction. The assertions that $(\bar{q})^{-1}(U) \rightarrow q_W^{-1}(U)$ is an isomorphism of schemes and that $q_W^{-1}(U) \rightarrow U$ is bijective are obvious for the generic quotient U asserted in (5) of §§2.2. \square

Lemma 2.3.4. *Let Δ be irreducible. Let $f: Q \rightarrow [X^{ss}/G]$ be the map (2.2.1) satisfying conditions (1)–(7) of §§2.2. Assume L_Q descends to a line bundle $L_{\bar{Q}}$ on \bar{Q} . If L is G -semi-ample on X^{ss} then*

- (1) L_Q is semi-ample.
- (2) If $R = \bigoplus_{n \geq 0} R_n$ is the graded ring whose n^{th} -graded piece is given by $R_n = \Gamma(X^{ss}, L^n)^G$, then R is a finitely generated k -algebra.

- (3) We have a morphism of schemes $\overline{Q} \rightarrow \text{Proj } R$ and a bijective continuous map $\text{Proj } R \rightarrow Y$ fitting into a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & \overline{Q} \\ q \downarrow & & \downarrow \\ Y & \longleftarrow & \text{Proj } R \end{array}$$

Proof. The assertion that L_Q is semi-ample is straightforward. Indeed, given a point $a \in Q$, we can find point $b \in P = X^{ss} \times_{[X^{ss}/G]} Q$ a positive integer n and a G -invariant section s of $L^n|_{X^{ss}}$ such that $s(\pi(b)) \neq 0$, whence the pull-back π^*s is non-vanishing on b . The corresponding section σ of L_Q^n is such that $\sigma(a) \neq 0$. From Lemma 2.3.1 it follows that $L_{\overline{Q}}$ is semi-ample. By replacing L by a suitable power if necessary, we assume that $L_{\overline{Q}}$ is actually base point free. We have the commutative diagram (2.3.2) with $W = \text{Proj } S$ as in (2.3.3). Note that $S_n = \Gamma(W, L_W^n)$.

Now $R \hookrightarrow S$. In fact R is a graded k -subalgebra of S . By our hypothesis, we can find a finite number of G - Γ -invariant sections of a suitable power L^n of L such that their pull backs to P considered as sections of powers of L_W^n have no base points. Let R' be the finitely generated k -algebra generated by these sections. Then $R' \rightarrow S$ is finite and since $R' \hookrightarrow R \hookrightarrow S$, R is a finite R' -algebra, whence a finitely generated k -algebra. Since $q_W: W \rightarrow Y$ has finite fibres, therefore for any $y \in Y$, $L_W|_{q_W^{-1}(y)}$ is trivial, whence the map $W \rightarrow \text{Proj } R$ sends $q_W^{-1}(y)$ to a single point in $\text{Proj } R$. It follows that we have a bijective continuous map $\text{Proj } R \rightarrow Y$. \square

Remark 2.3.5. Consider diagram (2.3.2). Since $\overline{Q} = Q/\Gamma$ and Γ is the group of $k(Y)$ automorphisms of $k(Q)$, it is clear that $k(Y) \rightarrow k(W) = k(\overline{Q})$ is a purely inseparable field extension.

Lemma 2.3.6. Let $Y^{(i)}$ be the distinct irreducible components of Y and $X^{(i)} = \alpha^{-1}(Y^{(i)})$ (with its reduced structure) $1 \leq i \leq r$. Suppose that L is G -semi-ample on $X^{(i)}$ for $1 \leq i \leq r$. Then L is G -semi-ample on X .

Proof. Let us note the following general fact. Let S be a projective variety and M an ample line bundle on S . Let T be a closed subscheme of S with ideal sheaf $I = I(T)$. Then given a section s of $M|_T$, s^r can be extended to a section of M^r for $r \gg 0$. This follows from the fact $H^1(S, M^r \otimes I) = 0$ for $r \gg 0$ and writing the usual exact sequence.

The lemma is proved by induction on r and hence we suppose that it is true for $X' = X^{(1)} \cup \dots \cup X^{(r-1)}$ (scheme theoretic union). Let $X'' = (X')^{ss} \cap (X^{(r)})^{ss}$. Write L_i for $L|_{X^{(i)}}$ and L'' for $L|_{X''}$. Then the image of X'' in $Y^{(r)}$ is a proper closed subset of $Y^{(r)}$. Let s be a G -invariant section, say of L'' and s_1 its restriction to X''_{red} . Let $R = R(X^{(r)})$ be the graded ring such that $R_n = \Gamma(X^{(r)}, L_r^n)^G$. Note

that $\text{Proj } R = Y^{(r)}$ as a topological space. Write L_R for the natural ample line bundle on $\text{Proj } R$ defined L . Then the image of X'' in $\text{Proj } R$ is a closed subscheme T of $\text{Proj } R$ with the reduced structure. Then we see that s_1 can be identified with a section of $L_R|_T$ (to do this apply Lemma 2.3.4). Then by the initial remark s_1 can be extended to a section of L_R on $\text{Proj } R$ (we may have to take a power of L and also raise s , to the same power) and we identify this extension as a G -invariant section t of L on $(X^{(r)})^{ss}$. Thus we see that s and t coincide on $(X'')_{\text{red}}^{ss}$ and then raising to the power p^a , $a \gg 0$, we can suppose by Lemma 3.6.5 that s and t coincide on X'' . Thus without loss of generality we can say that s can be extended to a G -invariant section on X^{ss} . Besides, we can also achieve this extension so that it does not vanish at a point of $Y^{(r)}$ outside T . From this the lemma follows easily \square

We will need the following result of Keel (see [SK, p. 254, Theorem 0.2]):

Theorem 2.3.7. *Let S be a projective variety of dimension n over an algebraically closed field of characteristic $p > 0$.³ Let L be a nef and big line bundle on S . Suppose that for any proper closed subset T of S , $L|_T$ is semi-ample for the reduced scheme structure on T . Then L is semi-ample on S .*

Proof. This is Keel's work already mentioned in the introduction. In greater detail, let S be any projective scheme and L a nef line bundle on S . An irreducible subvariety E of S is called *exceptional* if $L|_E$ is not big. The *exceptional locus* $E(L)$ of L is defined to be the closure with the reduced structure of the union of all the exceptional varieties. Then the result in [SK, p. 254, Theorem 0.2] is that L is semi-ample on S if and only if $L|_{E(L)}$ is semi-ample. \square

Theorem 2.3.8. *L is G -semi-ample on X^{ss} .*

Proof. By Lemma 2.3.6 it is enough to prove the theorem when Y is irreducible. We prove this by induction on $\dim Y$. In other words we suppose that if Δ^* is an irreducible strong quotient data with $\dim Y_{\Delta^*} < \dim Y$, then L_{Δ^*} is G -semi-ample on X_{Δ^*} . By Lemma 2.3.6 it follows that if Y' is any proper closed subset of Y , and if $X' = \alpha^{-1}(Y')$ is endowed with the reduced structure of a closed subscheme of X , then $L|_{X'}$ is G -semi-ample on X' .

Since Δ has been assumed to be irreducible, we have a normal irreducible projective variety Q and a map $f: Q \rightarrow [X^{ss}/G]$ satisfying (1)–(7) in §§ 2.2.

By the induction hypothesis, it follows that L_Q restricted to any proper closed subvariety of Q is semi-ample (we need property (7) of §§ 2.2). It follows then, by

³The reader is reminded that this is the default assumption on our underlying field k throughout the paper. We have re-stated this assumption in this statement, to emphasize that this is where the assumption of positive characteristic enters in a crucial way in this paper.

Theorem 2.3.7 that L_Q is semi-ample on W . Lemma 2.3.1 can therefore be applied. Consider the resulting commutative diagram (2.3.2), namely:

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & \overline{Q} \\ q \downarrow & & \downarrow \psi \\ Y & \xleftarrow{q_W} & W \end{array}$$

According to *loc.cit.* we have a generic quotient U for Δ such that, with $q' = q_w \circ \psi$, $(q')^{-1}(U) \rightarrow U$ and $q_W^{-1}(U) \rightarrow U$ are bijective continuous maps. Let $C = Y \setminus U$ and $Z = q_W^{-1}(C)$. Set $X_C^{ss} = \alpha^{-1}(C)$ and let X_C be its closure in X . Give Z and X_C^{ss} their reduced scheme structures and write $L_Z = L_W|_Z$. According to Lemma 2.3.4 C exists as a reduced scheme, and $L|_{X_C^{ss}}$ descends to an ample bundle L_C on C . Clearly L_Z descends to L_C . Let $q_Z: Z \rightarrow C$ be the resulting map induced by q_W . Consider the diagram, with the solid arrows already defined:

$$(2.3.9) \quad \begin{array}{ccc} Z & \xrightarrow{i} & W \\ q_Z \downarrow & & \downarrow g \\ C & \xrightarrow{i'} & \overline{W} \end{array}$$

We claim we can find a variety \overline{W} whose underlying topological space is Y , such that the dotted arrows in (2.3.9) can be filled to make the diagram commute, and such that L_W descends to an ample line bundle $L_{\overline{W}}$ on \overline{W} with the property that $L_{\overline{W}}|_C = L_C$.

The above claim is an easy consequence of the fact that for large positive n sections of L_Z^n which come from sections of L_C^n can be extended to sections of L_W^n such that these extensions separate points of $W \setminus Z = q_W^{-1}(U)$. For, if such extensions exist, they give a base point free linear system on W , which is a sub-linear system of the very ample linear system given by L_W^n . We now prove the existence of such extensions of such sections of $L_Z = q_Z^* L_C$. Let $x_1, x_2 \in q_W^{-1}(U)$, with $x_1 \neq x_2$, and set

$$Z' = Z \cup \{x_1, x_2\}$$

with Z' being given the reduced scheme structure. Let I be the ideal sheaf of the closed subscheme Z' of W . Let $L_{Z'} = L_W|_{Z'}$. Then we have an exact sequence of coherent $\mathcal{O}_{W'}$ -modules

$$0 \rightarrow L_W^n \otimes I \rightarrow L_W^n \rightarrow L_{Z'}^n \rightarrow 0.$$

Let $\sigma \in \Gamma(C, L_C^n)$. The section $q_Z^*(\sigma) \in \Gamma(Z, L_Z^n)$ can be extended to a section s of $L_{Z'}^n$, by setting $s(x_1) \neq 0$ and $s(x_2) = 0$ (for $L_{Z'}^n = L_Z^n \oplus k_{x_1} \oplus k_{x_2}$, where k_{x_i} is the residue field of \mathcal{O}_{W, x_i} for $i = 1, 2$). Since $H^1(W, L_W^n \otimes I) = 0$ for $n \gg 0$, the short

exact sequence of sheaves displayed above shows that s can be extended to all of W to give a section of L_W provided n is large enough.

Clearly the line bundle $L_{\overline{W}}$ is ample on \overline{W} and the underlying topological space of \overline{W} is Y . It is now evident that the dotted arrows in (2.3.9) can be filled as required.

Since $P(= Q \times_{[X^{ss}/G]} X^{ss})$ is irreducible therefore there exists a unique component X_1^{ss} of X^{ss} on which $p: P \rightarrow X^{ss}$ takes values. Let \tilde{X}_1^{ss} and \tilde{L}_1 denote, respectively, the normalization of X_1^{ss} and the pull back of L to \tilde{X}_1^{ss} . Note that the map $\tilde{p}: P \rightarrow \tilde{X}_1^{ss}$ actually takes values in \tilde{X}_1^{ss} . Let $s \in \Gamma(\overline{W}, L_{\overline{W}}^n)$. Then its pull-back $s^* \in \Gamma(W, L_W^n)$ arises from a G -invariant section $\sigma_s \in \Gamma(\tilde{X}_1^{ss}, \tilde{L}_1^n)^G$, and whence identifies with a regular function on the normalization of the cone \hat{X}_1^{ss} over X_1^{ss} . The graph of this section defines a closed subset Γ of $\hat{X}_1^{ss} \times_k \mathbb{A}^1$. Since σ_s arises from $s \in \Gamma(\overline{W}, L)$, which is topologically Y , the projection $\Gamma \rightarrow \hat{X}_1^{ss}$ is proper and bijective. By Lemma 3.6.5 suitable power of p of this function on the normalization goes down to a regular function on \hat{X}_1^{ss} . Yet another power of this section extends to X^{ss} . Such sections achieve the required G -semi-ampleness. \square

Corollary 2.3.10. *Let $S = \oplus_{n \geq 0} S_n$ be the graded ring defined by $S_n = \Gamma(X^{ss}, L^n)$. Then $R := S^G$ is a finitely generated graded ring, and Y acquires a canonical scheme structure $Y = \text{Proj } R$.*

Proof. By Lemma 2.3.4, this follows when X is irreducible. Then by a devissage argument as in [S3], the corollary follows. One formulates a more general statement that if M is a G -coherent module on $\mathbb{P}(V)$ then M^G is finitely generated. We leave the details to the reader. \square

3. Basic properties of Semi-stable equivalence

Throughout this section, we fix a G -triple $(X, L, \mathbb{P}(V))$. In this section we study the space Y of semi-stable equivalence classes as a topological space and show that it has the expected properties which are consequences of geometric reductivity. For example, granting the notions of semi-stability, unstability etc., as given below, if X^{ss} denotes the semi-stable locus of X with respect to L , then in Proposition 3.6.3 and Corollary 3.6.8 we show, respectively, that the graph of the semi-stable equivalence relation in $X^{ss} \times_k X^{ss}$ is closed and that it is “proper” i.e. “ $X^{ss} \text{ mod } G$ is proper”. As a consequence, Y is *separated*—i.e. the diagonal map $Y \rightarrow Y \times Y$ is closed with respect to the topology on $Y \times Y$ induced by the action of $G \times_k G$ on $X^{ss} \times_k X^{ss}$ —and Y is “proper”. Moreover, in Proposition 3.6.4 we show that a suitable power of L , when restricted to a semi-stable equivalence class, is trivial. This is the first step towards seeing that a power of L “descends to Y ”.

The key result for all these is Proposition 3.4.1, i.e. that the “ S -unstable locus” $\mathcal{U}(S)$ is closed in X^{ss} (for suitable G -stable subsets S of X^{ss}). This corresponds to the expected result that for the quotient map $\alpha : X^{ss} \rightarrow Y$, $\alpha(S)$ is closed. An important technical point concerns extensions of functions by taking their p -th powers Lemma 3.6.5. This is probably well-known to experts but we give a proof here as it is an important point where $\text{char } k = p > 0$ is used.

3.1. Semi-stability, polystability, stability, and instability. Recall that if $(X, L, \mathbb{P}(V))$ is a G -triple, then a point \hat{x} of \hat{X} is said to be *unstable* if the vertex $0 \in \hat{X}$ of the cone \hat{X} lies in the orbit closure \widehat{xG} . A point on \hat{X} which is not unstable is called *semi-stable*. The locus of semi-stable points is denoted by \hat{X}^{ss} . A point $x \in X$ is said to be semi-stable if for some $\hat{x} \in \hat{X} \setminus \{0\}$ lying over x , \hat{x} is semi-stable. Since the homothety action on \hat{X} commutes with the G -action on \hat{X} , $x \in X$ is semi-stable if and only if every point $\hat{x} \in \hat{X} \setminus \{0\}$ lying over x is semi-stable. The semi-stable locus in X is denoted $X^{ss}(L)$, or, if the line bundle L is understood from the context, simply X^{ss} .

A point $\hat{x} \in \hat{X}$ is said to be *polystable* if $\hat{x} \neq 0$ and the G -orbit through \hat{x} is closed. We say $x \in X$ is polystable if for some (and hence all) $\hat{x} \in \hat{X} \setminus \{0\}$ lying over x , \hat{x} is polystable.

We say $\hat{x} \in \hat{X}$ is *stable*, or *properly stable*, if the G -orbit \hat{xG} is closed in \hat{X} and $\dim \hat{xG} = \dim G$. Equivalently, $\hat{x} \in \hat{X}$ is stable if the orbit morphism $G \rightarrow \hat{X}$, $g \mapsto \hat{x}g$ is proper. We denote by \hat{X}^s the G -stable locus of stable points in \hat{X} . A point $x \in X$ is said to be stable if for some (and hence all) $\hat{x} \in \hat{X} \setminus \{0\}$ lying over x , \hat{x} is stable. We denote by $X^s(L)$ (or, if L is understood from the context, by simply X^s) the locus of stable points in X . Note that a stable point is polystable.

The notion of an unstable point can be generalized as follows:

Definition 3.1.1. Let $(X, L, \mathbb{P}(V))$ be a G -triple. Let S be a closed G -invariant subset of \hat{X} (e.g. $\hat{X} = V$), which we can endow with the reduced structure. Following Kempf [GK], we say that a point $\hat{x} \in \hat{X}$ is S -unstable if the orbit closure \widehat{xG} meets S . We denote by $\mathcal{U}(S, \hat{X}) = \mathcal{U}(S)$ the set of S -unstable points in \hat{X} . If $\lambda : \mathbb{G}_m \rightarrow G$ is a 1-PS of G , then the locus of S -unstable points in \hat{X} under the action of λ on \hat{X} is denoted $\mathcal{U}(S, \lambda) = \mathcal{U}(S, \hat{X}, \lambda)$.

Remarks 3.1.2.

- (a) Note that a point in \hat{X} is (0)-unstable if and only if it is unstable.
- (b) $\mathcal{U}(S, \hat{X}) = \mathcal{U}(S, V) \cap \hat{X}$.
- (c) The set $\mathcal{U}(S)$ is again G -stable, and $S \subset \mathcal{U}(S)$.
- (d) if S is homogeneous (i.e., invariant under the the homothety action), then $\mathcal{U}(S)$ is again homogeneous.

3.2. The μ function. We shall now recall some basic facts from Geometric Invariant Theory [M]. Critical to our understanding stability, semi-stability, and unstability is the notion of the μ -function. We give two definitions : Definition 3.2.1 as well as a more intrinsic one (applicable in more general situations) in Definition 3.2.3.

Let $(X, L, \mathbb{P}(V))$ be a G -triple such that the standard linear system on $\mathbb{P}(V)$ arising from $\mathcal{O}(1)$, when restricted to X , is a complete linear system (i.e., the *trace* on X —of the complete linear system on $\mathbb{P}(V)$ —is a complete linear system on X). This is equivalent to saying that $H^0(X, L) = V^*$. Let $\lambda: \mathbb{G}_m \rightarrow G$ be a 1-PS. Recall that the action of λ on $X, \mathbb{P}(V), \widehat{X}, V$ is, by definition, the action of \mathbb{G}_m on these spaces induced by λ and the G -action on them (see (6) of Subsection 1.1).

Now, action of λ on V can be diagonalised, whence we can find a basis $\{e_i\}$ of V such that the one dimensional subspaces ke_i are \mathbb{G}_m -stable under the action of λ , and hence give rise to characters, one for each subscript i :

$$\chi_i: \mathbb{G}_m \rightarrow \mathbb{G}_m, \quad t \mapsto t^{r_i}$$

with $r_i = r_i^L(\lambda)$ an integer. Now, suppose $x \in X$ and suppose

$$\widehat{x} = \sum x_i e_i$$

is a point of $\widehat{X} \setminus \{0\}$ lying over x .

Definition 3.2.1. (μ -function) With notations as above, the μ -value of x with respect to L and λ is

$$\mu^L(x, \lambda) := \max_{j, x_j \neq 0} \{-r_j\} = - \min_{j, x_j \neq 0} \{r_j\}.$$

Clearly, the definition of $\mu^L(x, \lambda)$ does not depend on the choice of the point \widehat{x} lying above x . If the role of L is understood from the context, we will simply write $\mu(x, \lambda)$ for $\mu^L(x, \lambda)$.

Note that we have:

$$(3.2.2) \quad \begin{aligned} \lim_{t \rightarrow 0} x \circ \lambda \text{ exists} &\iff \mu^L(x, \lambda) \geq 0 \\ \lim_{t \rightarrow 0} x \circ \lambda = 0 &\iff \mu^L(x, \lambda) > 0 \end{aligned}$$

We can define the function μ in a more intrinsic manner as follows: Let G act on a *complete* k -scheme X , and suppose the action lifts linearly to an action on a line bundle L on X . Note that we are not assuming that X is projective, or that L is ample. Let $x \in X$, λ a 1-PS, and denote by $\Psi_x: \mathbb{G}_m \rightarrow X$ the orbit morphism defined by $t \mapsto x \circ \lambda(t)$. Since X is complete, the morphism Ψ_x extends to a morphism of \mathbb{P}^1 into X , which we again denote by Ψ_x . Let $x_\circ = \Psi_x(0)$. Then x_\circ is invariant under the action of λ . Let L_\circ be the fibre of $L \rightarrow X$ over x_\circ . Then the operation of \mathbb{G}_m on L_\circ is defined by a character $\chi: \mathbb{G}_m \rightarrow \mathbb{G}_m$ defined by $t \mapsto t^r$.

Definition 3.2.3. (Intrinsic definition of μ) Let X , L , λ , x , and r be as above. Then the μ -value of x with respect to L and λ is

$$\mu^L(x, \lambda)(= \mu(x, \lambda)) = -r.$$

3.3. The Hilbert-Mumford criterion for S -unstability. Let $(X, L, \mathbb{P}(V))$ be a G -triple. Suppose $S \subset V$ is a G -invariant closed subscheme. We claim that we can find a finite dimensional rational G -module W and a G -equivariant morphism

$$(3.3.1) \quad f: V \rightarrow W$$

(now thinking of V and W as schemes) such that set-theoretically, or even scheme-theoretically, $f^{-1}(0) = S$. Indeed, let F_1, \dots, F_l be generators of the ideal $I_S \subset k[V]$ of S , where $k[V]$ is the ring of functions $\Gamma(V, \mathcal{O}_V)$ on the affine scheme V . In other words, $k[V] = S(V^*)$, the symmetric algebra on the dual linear space V^* of V . By considering—if necessary—the G -span of the k -linear space spanned by F_1, \dots, F_l , we may assume that the linear space spanned by the F_i is G -stable and that F_1, \dots, F_l are linearly independent over k . Let W^* be this l -dimensional G -stable subspace of $k[V]$, and let W be its dual. Now, $S(W^*)$ maps in a *surjective* and *G -equivariant* manner onto the subalgebra $k[F_1, \dots, F_l]$ of $k[V]$, and we have a sequence of G -equivariant maps of k -algebras:

$$k[W] = S(W^*) \twoheadrightarrow k[F_1, \dots, F_l] \subset k[V]$$

The resulting algebraic map of varieties $f: V \rightarrow W$ meets our requirements.

We would like to understand the locus of points in $\mathcal{U}(S) \setminus S$. Suppose $v \in \mathcal{U}(S) \setminus S$. We can find a discrete valuation ring (d.v.r) A with residue field k and quotient field K , as well as K valued point of G , $\theta \in G(K)$, such that $v \circ \theta \in V(K)$ is really an A -valued point, i.e., $v \circ \theta: A \rightarrow V$, and the closed point of A maps to a k -valued point $v_\circ \in S$. If t denotes the 1-parameter represented by $\text{Spec } A$, we write, as a shorthand,

$$\lim_{t \rightarrow 0} v \circ \theta(t) = v_\circ.$$

In this situation, the Hilbert-Mumford Theorem, as generalized by Kempf [GK] for the case of S -instability, states that there exists in fact, a 1-PS $\lambda: \mathbb{G}_m \rightarrow G$ such that (in an obvious notation)

$$\lim_{t \rightarrow 0} v \circ \lambda(t) = v_\circ.$$

One can show:

Theorem 3.3.2. (Hilbert-Mumford) *Let $(X, L, \mathbb{P}(V))$ be a G -triple and $S \subset \widehat{X}$ a closed G -invariant subset. Let $f: V \rightarrow W$ be as in (3.3.1), i.e., f is a map of varieties such that $f^{-1}(0) = S$. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$ of G , and let $\Gamma(T)$ be the co-root lattice of T and $C(B)$ the Weyl chamber*

in $\Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$ corresponding to B , and $\overline{C(B)}$ its closure in the Euclidean space $\Gamma(T) \otimes \mathbb{R}$.

(a) The following are equivalent:

- (i) $v \in \mathcal{U}(S) \setminus S$;
- (ii) $f(v) \neq 0$ and there exists a one-parameter subgroup λ of G such that $\lim_{t \rightarrow 0} v \circ \lambda(t)$ exists and $\lim_{t \rightarrow 0} f(v) \circ \lambda(t) = 0$;
- (iii) $f(v) \neq 0$ and there exists $g \in G$ and a 1-PS λ of T in the chamber $\overline{C(B)}$ such that $\lim_{t \rightarrow 0} (v \circ g)\lambda(t)$ exists and $\lim_{t \rightarrow 0} f(v \circ g)\lambda(t) = 0$.

(b) The following are equivalent:

- (iv) $v \in \mathcal{U}(S)$;
- (v) There exists a one-parameter subgroup λ of G such that $v \in \mathcal{U}(S, \lambda)$;
- (vi) There exists $g \in G$ and a one-parameter subgroup λ of T in the chamber $\overline{C(B)}$ such that $v \circ g \in \mathcal{U}(S, \lambda)$.

Proof. In view of the comments made before the statement of the theorem, evidently, (i) and (ii) are equivalent as are (iv) and (v). For the rest we only need to recall the well-known fact that every 1-PS λ of G is conjugate to a 1-PS in T , in fact to an integral point in $\overline{C(B)}$. \square

Proposition 3.3.3. *Let the notations be as in the hypotheses of Theorem 3.3.2. There exist a finite number of 1-PS $\lambda_1, \dots, \lambda_N$ of T in the chamber $\overline{C(B)}$ such that $\mathcal{U}(S) = \bigcup_{1 \leq i \leq N} \mathcal{U}(S, \lambda_i) \cdot G$.*

Proof. According to Lemma 5.1.1 in Section 5 below, there are a finite number of closed convex cones C_α in $\overline{C(B)}$ —each C_α an intersection of a finite number closed half-spaces—such that for every $v \in V$ and $w \in W$, $\mu(v, -)$ and $\mu(w, -)$ are linear in each C_α . This is seen setting $d = 2$, $X_1 = \mathbb{P}(V)$, and $X_2 = \mathbb{P}(W)$ in *loc.cit.* Let \mathfrak{S}_α be a finite set of generators over \mathbb{R}^+ for the cone C_α . We can choose \mathfrak{S}_α with integral coefficients, i.e., $\mathfrak{S}_\alpha \subset \Gamma(T)$. Let $\mathfrak{S} = \bigcup_\alpha \mathfrak{S}_\alpha$. Since $\mu(v, -)$ and $\mu(f(v), -)$ are linear in each C_α , we conclude that:

$\mu(v, \lambda) \geq 0$ and $\mu(f(v), \lambda) > 0$ for every $\lambda \in \overline{C(B)} \setminus \{0\}$ if and only if $\mu(v, \lambda) \geq 0$ and $\mu(f(v), \lambda) > 0$ for every $\lambda \in \mathfrak{S}$.

Let $\mathfrak{S} = \{\lambda_1, \dots, \lambda_N\}$. Then by Theorem 3.3.2(a)—especially (i) \Leftrightarrow (iii)—and the relations (3.2.2), it is evident that $\mathcal{U}(S)$ is the union of $\mathcal{U}(S, \lambda_i) \cdot G$ for $i = 1, \dots, N$. \square

3.4. The S -unstable locus $\mathcal{U}(S)$ is closed. Fix a G -triple $(X, L, \mathbb{P}(V))$. Recall the following properties of the function $\mu = \mu^L$ (see [S2, Prop. 3.1]): Let $\lambda: \mathbb{G}_m \rightarrow T$ be a 1-PS and let $P(\lambda)(\supset B)$ be the parabolic subgroup of G defined by

$$P(\lambda) := \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)^{-1} g \lambda(t) \text{ exists in } G\}.$$

Then, for every $v \in V$, we have:

- (a) (Change of coordinates) $\mu(v, \lambda) = \mu(v \circ g, g\lambda g^{-1})$, $g \in G$.
- (b) $\mu(v, \lambda) = \mu(v, g\lambda g^{-1})$, $g \in P(\lambda)$.
- (c) $\mu(v, \lambda) = \mu(v \circ g, \lambda)$, $g \in P(\lambda)$.
- (d) $\mu(v, \lambda) = \mu(v \circ g, \lambda)$, $\lambda \in \overline{C(B)}$ and $g \in B$.

Note that (c) follows from (a) and (b), and (d) is a special case of (c), since $\lambda \in \overline{C(B)}$ implies that $B \subset P(\lambda)$.

Proposition 3.4.1. (See [S2, p. 524, Theorem 3.1]) *Let S be a G -invariant closed subset of \widehat{X} . Then the set $\mathcal{U}(S, \widehat{X})$ is a closed G -invariant subset of \widehat{X} .*

Proof. First note that for any 1-PS λ of G , the set $\mathcal{U}(S, \lambda)$ is closed. Indeed, since \mathbb{G}_m is linearly reductive, the GIT quotient $V//\lambda$ for the action of λ on V exists as a variety and if $j: V \rightarrow V//\lambda$ is the canonical quotient morphism, then it is easily seen that $j(S)$ is closed in $V//\lambda$. Now, $\mathcal{U}(S, \lambda) = j^{-1}(j(S))$, thus proving that $\mathcal{U}(S, \lambda)$ is closed in V . Let $\lambda_1, \dots, \lambda_N$ be the finite number of 1-PS of T in $\overline{C(B)}$ guaranteed by Proposition 3.3.3 above, with the property that $\mathcal{U}(S) = \bigcup_{1 \leq i \leq N} \mathcal{U}(S, \lambda_i) \cdot G$. Each $\mathcal{U}(S, \lambda_i)$ is B -stable and hence so is

$$U = \bigcup_i \mathcal{U}(S, \lambda_i).$$

Now for any scheme Z on which B acts, one defines $Z \times^B G$ as the set of equivalence classes of the equivalence relation on $Z \times G$ given by $(z, g) \sim (v \circ b, b^{-1}g)$, $z \in Z$, $b \in B$, $g \in G$.⁴ The natural map $\pi_Z: Z \times^B G \rightarrow B \backslash G$ is a fibre bundle with fibre Z , and structure group B . The associated principal fibre bundle is the canonical quotient $G \rightarrow B \backslash G$. The subset $U \cdot G = \mathcal{U}(S)$ of V is the image of the map $U \times G \xrightarrow{f} V$ given by $(u, g) \mapsto u \circ g$. Let $p: U \times G \rightarrow U \times^B G$ be the natural quotient map. Then clearly the map $U \times G \xrightarrow{f} V$ factors through p , giving us a map $f': U \times^B G \rightarrow V$ such that $f = f' \circ p$, i.e., we have a commutative diagram:

$$\begin{array}{ccc} U \times G & \xrightarrow{p} & U \times^B G \\ & \searrow f & \downarrow f' \\ & & V \end{array}$$

Next note that the map $U \times^B G \rightarrow V$ given by $(v, g) \mapsto v \circ g$ is isomorphic to the trivial bundle $V \times B \backslash G \rightarrow V$, the explicit isomorphism $U \times^B G \xrightarrow{\sim} V \times B \backslash G$ being $(v, g) \mapsto (v \circ g, Bg)$. If $U \times^B G \hookrightarrow V \times^B G$ is the closed immersion induced

⁴The equivalence relationship is by a free group action of B , and hence the quotient $Z \times^B G$ exists as a scheme, and the quotient map $Z \times_k B \rightarrow Z \times^B G$ is a principal B -bundle.

by the closed immersion $U \hookrightarrow V$, we get a commutative diagram:

$$\begin{array}{ccccc}
 U \times G & \xrightarrow{p} & U \times^B G & \hookrightarrow & V \times^B G \\
 & \searrow f & \downarrow f' & & \parallel \\
 & & V & \xleftarrow{\text{projection}} & V \times B \backslash G
 \end{array}$$

Now $\mathcal{U}(S) = U \cdot G = f(U \times_k G) = f'(U \times^B G)$ —the last equality from the surjectivity of p . The map f' is clearly proper, being the composite of a closed immersion followed by the proper map $V \times_k B \backslash G \rightarrow V$. Thus $\mathcal{U}(S)$ is a closed subset of V . \square

Corollary 3.4.2. *The set \hat{X}^{ss} (resp. X^{ss}) is open and G -invariant in \hat{X} (resp. X). A point $x \in X^{ss}$ is stable i.e. $x \in X^s$ if and only if the orbit $O(x) = xG$ is closed in X^{ss} and $\dim O(x) = \dim G$ i.e. the orbit morphism $G \rightarrow X^{ss}$, $g \mapsto x \circ g$, is proper.*

Proof. Taking $S = (0)$ the first assertion follows. Recall that for $x \in \hat{X}^{ss}$, we have $\overline{O(x)}$ (closure in \hat{X}) $\subset \hat{X}^{ss}$. The second assertion of the corollary reduces to proving that for $x \in X^{ss}$, $O(x)$ closed in $X^{ss} \iff O(\hat{x})$ closed in \hat{X}^{ss} . But this is an immediate consequence of the following claim:

Let x be a point in X^{ss} and \hat{x} a point in \hat{X}^{ss} lying over x . Then for any $y \in \overline{O(x)} \cap X^{ss}$, there exists a point $\hat{y} \in \overline{O(\hat{x})}$ lying over y .

The claim is proved as follows: Set $A := k[[t]]$ and $K := k((t))$. Let C be an irreducible curve in X joining x and y and C_1 an irreducible curve in $\hat{X} \times G$ through (\hat{x}, e) mapping dominantly to C via $\hat{X} \times G \xrightarrow{\sigma} \hat{X} \rightarrow X$. Projecting C_1 to G results in a $\text{Spec}(K)$ -valued point g (or $g \in G(K)$), such that $xg \in X(A)$. As in the proof of [S2, p. 520, Theorem 2.2], we may assume that $\hat{X} = V$ (i.e., affine n -space), and that $g = U\lambda$ where $U \in G(A)$ satisfies $\lim_{t \rightarrow 0} U = \text{identity matrix}$ and $\lambda \in T(K)$ (T a maximal torus in G) is diagonal of the form $(t^{r_1}, \dots, t^{r_n})$.

It suffices to show that $\lim_{t \rightarrow 0} \hat{x}g$ exists, as then we may choose \hat{y} to be this limit point (which is necessarily nonzero owing to semistability of x). If the limit does not exist, then there is a unique integer $s < 0$ such that $\hat{z} := \lim_{t \rightarrow 0} t^{-s} \hat{x}g$ exists and is nonzero. Let \hat{z}_i denote the i -th coordinate of \hat{z} . Since $\{i \mid r_i \geq 0\} \subset \{i \mid \hat{z}_i = 0\}$, we see that \hat{z} is unstable via the action of λ^{-1} . Since \hat{z} lies over y , this contradicts semi-stability of y . \square

Definition 3.4.3. Let S be a closed G -stable subset of X^{ss} . A point $x \in X^{ss}$ is said to be S -unstable if the closure \overline{xG} in X^{ss} of the orbit xG meets S . We denote by $\mathcal{U}(S) = \mathcal{U}(X, S)$ the G -invariant subset of S -unstable points in X^{ss} .

Corollary 3.4.4. *With above notations, $\mathcal{U}(S)$ is a closed G -invariant subset of X^{ss} .*

Proof. Let \widehat{S} be the “cone over S ” i.e. the homogeneous closed subset of \widehat{X} defined by S . In view of the above lemma, the corollary is an easy consequence of the proposition applied to \widehat{S} . \square

Definition 3.4.5. Let $v_1, v_2 \in V$ (or \widehat{X}). We say write $v_1 \sim v_2$ if $\overline{O(v_1)} \cap \overline{O(v_2)} \neq \emptyset$ (i.e., the orbit closures of v_1 and v_2 intersect). Similarly for $x_1, x_2 \in X^{ss}$, we write $x_1 \sim x_2$ if $\overline{O(x_1)} \cap \overline{O(x_2)} \neq \emptyset$. One checks easily that $x_1 \sim x_2$ if and only if there exist $v_1, v_2 \in V \setminus \{0\}$ lying over x_1 and x_2 respectively such that $v_1 \sim v_2$. We call these relations *semi-stable (or orbit closure) equivalence relations*, on account of the following:

Corollary 3.4.6. *The relations defined in Definition 3.4.5 are equivalence relations.*

Proof. To prove the equivalence relation property, we see that it suffices to prove the transitivity property. We first observe that this property is equivalent to showing that there is a unique closed orbit in $\overline{O(x)}$ (closure of $O(x)$ for $x \in V$. It suffices to consider this case). To see this, suppose that $\overline{O(x)}$ contains two distinct closed orbits $\overline{O(x_1)}$ and $\overline{O(x_2)}$. Then we have $x_1 \sim x$ and $x \sim x_2$ but x_1 is not equivalent to x_2 . On the other hand if every orbit closure has a unique closed orbit then we see that $x_1 \sim x_2$ if and only if the unique closed orbits in $\overline{O(x_1)}$ and $\overline{O(x_2)}$ coincide. Then the transitivity property is immediate. Let us now prove that every $\overline{O(x)}$ has a unique closed orbit. Again suppose that there are two distinct closed orbits $\overline{O(x_1)}$ and $\overline{O(x_2)}$ in $\overline{O(x)}$. Then we see that x is $\overline{O(x_1)}$ -unstable, which implies that $\overline{O(x)} \subset \mathcal{U}(\overline{O(x_1)})$ (by the proposition). But $x_2 \notin \mathcal{U}(\overline{O(x_1)})$, which leads to a contradiction. This proves that the relation is transitive. \square

Corollary 3.4.7. *Let \widehat{X}^e and $\widehat{X}^{ps,e}$ denote the subsets of \widehat{X} defined by:*

$$\begin{aligned}\widehat{X}^e &= \{x \in \widehat{X} \mid \dim O(x) \leq e\} \\ \widehat{X}^{ps,e} &= \{x \in \widehat{X} \mid \dim O(x) = e \text{ and } x \text{ is polystable}\}.\end{aligned}$$

Similarly, let $X^{ss,e}$ and $X^{ps,e}$ denote the subsets of X^{ss} defined by

$$\begin{aligned}X^{ss,e} &= \{x \in X^{ss} \mid \dim O(x) \leq e\} \\ X^{ps,e} &= \{x \in X^{ss} \mid x \text{ is polystable and } \dim O(x) = e\}\end{aligned}$$

Then

- (1) \widehat{X}^e is closed and G -invariant in \widehat{X} .
- (2) $\widehat{X}^{ps,e}$ is open and G -invariant in \widehat{X}^e .
- (3) $X^{ss,e}$ is closed and G -invariant in X^{ss} .
- (4) $X^{ps,e}$ is open and G -invariant in $X^{ss,e}$.

Observations: Specializing to $e = \dim G$ in (2), we have \widehat{X}^s is open and G -invariant in \widehat{X} . Note also that, with I_x the isotropy group at x , we have description:

$$\widehat{X}^e = \{x \in X \mid \dim I_x \geq \dim G - e\}.$$

The sets \widehat{X}^s and $\widehat{X}^{ps,e}$ could be empty.

Proof. That \widehat{X}^e is closed is seen easily. It is of course G -invariant.

$$\widehat{X}^e \setminus \widehat{X}^{ps,e} = \mathcal{U}(\widehat{X}^{e-1}) \text{ (in } \widehat{X}^e)$$

and then the corollary follows from the proposition. The proofs of the other assertions are similar. \square

3.5. Base Change for X^{ss} . Throughout this subsection we fix $(X, L, \mathbb{P}(V))$, which is a G -triple over k . Let k' be an extension of k and say k' is also algebraically closed. Let $X_{k'}$ be the base change of X by $\text{Spec } k' \rightarrow \text{Spec } k$ (similarly $\widehat{X}_{k'}$ etc.). Then we claim that $X_{k'}^{ss}$ is the base change of X^{ss} by $\text{Spec } k' \rightarrow \text{Spec } k$. When the group G is a torus, say a 1-PS subgroup, this assertion is seen easily. Then the general case follows from Proposition 3.3.3. The point of this proof can also be stated as follows. A point x of $X_{k'}$ i.e. an element of $X(k')$ (k' valued points of X) is semi-stable if and only if \exists a finite number of 1-PS $\lambda_1, \dots, \lambda_N$ defined over k in the chamber $C(B)$ and a point $g \in G(k')$ such that $x \circ g'$ is semi-stable with respect to λ_i . This reduces the assertion to the case of a 1-PS.

Suppose that the base field k is not algebraically closed, but the schemes and actions are defined over k . Let us define a geometric point x of X (i.e. $x \in X(\Omega)$, where Ω is algebraically closed) to be semi-stable if x is a semi-stable point of the base change X_Ω . Then the above argument, in fact, shows that there is a G -stable open subscheme X^{ss} of X such that its geometric points are precisely the semi-stable points. This should of course be stated for more general base schemes S . One should define semi-stability only for geometric points and have an open subscheme X^{ss} whose points are precisely the semi-stable points as in [S3].

Remark 3.5.1. Let A denote the discrete valuation ring $k[[t]]$ with quotient field K . Let x be a K -valued point of X_K , with X_K denoting base change, as above. Let H be an algebraic subgroup of G . Then H_K operates on X_K . Let Z_K be the closure in X_K of the H_K orbit through x . Let Z_A be the closed subscheme of X_A , flat over A , determined by Z_K . Then we see that the group scheme $H_A = H \times_k \text{Spec } A$ operates on Z_A and the generic fibre Z_K of $Z_A \rightarrow \text{Spec } A$ contains an open orbit under the action of H_K (namely the H_K orbit through x).

Lemma 3.5.2. [S2, pp. 528–529, Rmk. 4.9] *Let $x \in X^{ss}$ and Z the G -orbit through x with its reduced structure. Then there exists an integer $n > 0$ such that the restriction of L^n to Z is trivial.*

Proof. Let H be the isotropy subgroup of G at the point x . Then Z is the homogeneous space $H \backslash G$ and the restriction of the G -line bundle to Z is defined by a character $\chi : H \rightarrow \mathbb{G}_n$. It suffices to prove that $\exists n > 0$ such that χ^n is the trivial character. Suppose that this is not the case. Then we see easily that we can find a 1-PS λ of H such that $(\chi \circ \lambda) : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is surjective. We see that $(\chi \circ \lambda)$ is the character defining the action of λ on the fibre L at x and the surjectivity of $\chi \circ \lambda$ implies that $\mu(x, \lambda) \neq 0$ (see Definition 3.2.3 above). Then either $\mu(x, \lambda)$ or $\mu(x, \lambda)$ is < 0 . This contradicts the fact that $x \in X^{ss}$. \square

3.6. Separatedness and properness properties of $(X^{ss} // G)_{\text{top}}$. In this section we investigate some basic topological properties of the semi-stable equivalence relation defined in Definition 3.4.5 (see also Corollary 3.4.6). A crucial tool is Lemma 3.6.5 concerning proper bijective maps of algebraic schemes over fields of positive characteristic.

For the remainder of this section we fix a quotient data

$$(3.6.1) \quad \Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y).$$

We may assume, without loss of generality, if the occasion demands, that V is the vector space dual of $H^0(X, L)$, i.e., that the linear system on X induced by the trace of the tautological linear system on $\mathbb{P}(V)$ is complete.

Definition 3.6.2. The *product topology* on $Y \times Y$ is the quotient topology induced by the action of $G \times_k G$ on $X^{ss} \times_k X^{ss}$. In other words, as topological spaces

$$Y \times Y = (X^{ss} \times_k X^{ss} // (G \times_k G))_{\text{top}}.$$

We will be showing that Y is “separated”, and “complete” in Proposition 3.6.3 and Corollary 3.6.8 respectively.

Proposition 3.6.3. *The graphs of the semi-stable equivalence relations are closed. In fact, if Δ_X is the diagonal in $\widehat{X} \times \widehat{X}$ (or $X^{ss} \times X^{ss}$) the graph Γ of the equivalence relation is given by*

$$\Gamma = \mathcal{U}(\overline{\Delta_X \cdot (G \times G)}).$$

In particular Y is “separated”, i.e., the diagonal map $Y \rightarrow Y \times Y$ is a closed embedding, with $Y \times Y$ having the product topology defined above.

Proof. Let us work with the case $\widehat{X} = V$. The proofs in the other cases are immediate consequences. Let us take a d.v.r. A and its quotient K as in Remark 3.5.1. Note that $\overline{\Delta_X \cdot (G \times G)}$ consists of points (x_0, y_0) of the form:

- (i) x and y are A -valued points of V .
- (ii) $\lim_{t \rightarrow 0} x(t)(=x) = x_0, \lim_{t \rightarrow 0} y(t)(=y) = y_0$.
- (iii) There exists $g \in G(K)$ such that $y = x \cdot g$.

(iv) $x_0 \sim y_0$.

By Iwahori-Matsumoto [I-M] (possibly by going to the integral closure in a finite extension of A), we have $g = P\lambda Q$, where P, Q are in $G(A)$ and $\lambda : \mathbb{G}_m \rightarrow G$ is a 1-PS defined by a K -valued point of G . Set $x' = x \cdot P$ and $y' = x \circ P\lambda = x' \circ \lambda$. We see that $x'_0 = \lim_{t \rightarrow 0} x'(t)$, $y'_0 = \lim_{t \rightarrow 0} y(t)$ are in the G -orbits of x_0 and y_0 respectively (for $y_0 = y'_0 \cdot Q_0$, $Q_0 = \lim_{t \rightarrow 0} Q$) and thus to prove (iv) we can assume without loss of generality that $y = x \circ \lambda$, so that we are reduced to the case G is a torus (of dimension one). In this case we know that x_0 is semi-stably equivalent to y_0 for the action of λ , hence a fortiori $x_0 \sim y_0$.

Thus we see that $\overline{\Delta_X \cdot (G \times G)} \subset \Gamma$. Take a point $(x_0, y_0) \in \Gamma$. Then $\overline{O(x_0)}$ and $\overline{O(y_0)}$ have a unique common closed orbit $O(z)$ and we can suppose that $\exists g = g_t, h = h_t$ with $g, h \in G(K)$ and $x_0 \circ g, y_0 \circ h$ are A -valued points of V with $z = \lim_{t \rightarrow 0} x_0 \circ g = \lim_{t \rightarrow 0} y_0 \circ h$. This implies that $(x_0, y_0) \in \mathcal{U}(\overline{\Delta_X \cdot (G \times G)})$ so that $\Gamma \subset \mathcal{U}(\overline{\Delta_X \cdot (G \times G)})$. On the other hand let $(z, w) \in \mathcal{U}(\overline{\Delta_X \cdot (G \times G)})$. This means that $\exists g = g_t, h = h_t$, such that if $z_0 = \lim_{t \rightarrow 0} z \circ g_t$, and $w_0 = \lim_{t \rightarrow 0} w \circ h_t$ (in the sense as above) then $(z_0, w_0) \in \overline{(\Delta_X \cdot (G \times G))}$. Then as shown above $z_0 \sim w_0$. On the other hand we have obviously $z \sim z_0, w \sim w_0$. Hence by the transitivity of the semi-stable equivalence relation, we see that $z \sim w$ i.e. $\mathcal{U}(\overline{\Delta_X \cdot (G \times G)}) \subset \Gamma$. Thus we have $\Gamma = \mathcal{U}(\overline{\Delta_X \cdot (G \times G)})$. \square

The next Proposition, viz. Proposition 3.6.4 says that if the quotient Y is a single point, then L is essentially trivial. More precisely, a power of L is trivial. Morally then, L could be regarded as the pull back of a line bundle on Y , which, consisting of exactly one point, carries up to isomorphism, only one line bundle. However, Y not (as yet) having a scheme structure, these observations right now come under the heading of “gathering evidence” that Y has a scheme structure. The Proposition plays an essential role in showing that Y has a natural scheme structure, making it the GIT quotient Y .

Proposition 3.6.4. *Suppose the quotient data Δ is reduced and X^{ss} consists of one semi-stable equivalence class, i.e., any two points of X^{ss} are semi-stably equivalent. Suppose further that the closure of X^{ss} in $\mathbb{P}(V)$ is X . Let \hat{X} stand for the cone over X with its reduced structure. Then a suitable power of L has a section s over X which is G -invariant and non-vanishing for every $x \in X^{ss}$. In fact we can find s which comes from a regular G -invariant function on \hat{X} .*

Proof. We have a unique closed orbit in X^{ss} . We denote the closure of this in X by X_1 . Then we observe that

$$(i) \quad \mathcal{U}(\hat{X}_1) = \hat{X}$$

i.e. the closure of the G -orbit through any point of \widehat{X} meets \widehat{X}_1 . Let us now suppose that the proposition holds for X_1 (the proof will be given later after the proof of Lemma 3.6.6). We first show that the proposition is true under our supposition.

We have then a G -invariant function f on \widehat{X}_1 such that $f(x) \neq 0 \forall x \in \widehat{X}_1^{ss}$ (of course $f(x) = 0 \forall x \in \widehat{X} \setminus \widehat{X}^{ss}$) which is “homogeneous” i.e. $f(tx) = t^m f(x)$, where multiplication by t denotes the homothety action and L is the m th power of the tautological ample line bundle on $\mathbb{P}(V)$. Let us extend f to a set theoretic function F ($F : \widehat{X} \rightarrow \mathbb{A}^1$) as follows. Given $x \in \widehat{X}$, the orbit closure $\overline{O(x)}$ meets \widehat{X}_1 (by (i)), say at a point y . Let us set $F(x) = F(y)$. We claim that it is well-defined. For, if y_1, y_2 are two such points, $y_1 \sim y_2$. If $x \notin \widehat{X}^{ss}$, $y_1, y_2 \in X_1 \setminus X_1^{ss}$ so that $F(x) = f(y_1) = f(y_2) = 0$. If $x \in X^{ss}$, y_1, y_2 are semi-stably equivalent, which implies that $y_2 \in \overline{O(y_1)}$. Since f is G -invariant, it follows that $f(y_1) = f(y_2)$. Thus the set theoretic function F is well-defined, G -invariant and is an extension of f . We observe that F is also “homogeneous of the same degree” as that of f .

Note that if the complete reducibility property holds for G (e.g. G a torus), then one knows that f can be extended to a G -invariant function on \widehat{X} and in our case F is uniquely determined. Thus in this case $F : \widehat{X} \rightarrow \mathbb{A}^1$ is a morphism. We will use this fact below.

As we have seen in Proposition 3.3.3 there exist 1-PS λ_i , $1 \leq i \leq N$, of T in $\overline{C(B)}$ such that

$$(ii) \quad \widehat{X} = \bigcup_{1 \leq i \leq N} \mathcal{U}(\widehat{X}_1, \lambda_i) \cdot G.$$

Set $\mathcal{U}_i = \mathcal{U}(\widehat{X}_1, \lambda_i)$ i.e. \mathcal{U}_i is the set of \widehat{X}_1 unstable points for the 1-PS λ_i , then there is a unique λ_i -invariant regular function θ_i on \mathcal{U}_i which extends f . Now F coincides with θ_i on \mathcal{U}_i so that we see that the restriction to each \mathcal{U}_i is regular. Now $\theta_i - \theta_j$ vanishes set theoretically on the scheme theoretic intersection $\mathcal{U}_i \cap \mathcal{U}_j$. Hence

$$(iii)' \quad \begin{cases} (\theta_i - \theta_j)^q = (\theta_i^q - \theta_j^q) \text{ vanishes on the scheme intersection } \mathcal{U}_i \cap \mathcal{U}_j, \text{ for} \\ q = p^r, r \gg 0. \end{cases}$$

Let then \mathcal{U} denote the (scheme theoretic) union $\bigcup \mathcal{U}_i$. Then \mathcal{U} is reduced, \mathcal{U}_i being reduced. We see for $q = p^r$, $r \gg 0$, the restriction of F^q to \mathcal{U} is a regular function. Thus without loss of generality, we can assume that the restriction of F to \mathcal{U} is a regular function.

Let F' be the regular function on $\mathcal{U} \times G$ defined by $F'(xg) = F(x)$; $x \in \mathcal{U}$, $g \in G$. As we have seen in Proposition 1.1, the morphism $\mathcal{U} \times G \rightarrow \widehat{X}$, $(x, g) \mapsto x \circ g$, factors as follows $\mathcal{U} \times G \rightarrow \mathcal{U} \times^B G \rightarrow \widehat{X}$. We see that F' goes down to the function F on \widehat{X} , in particular it goes down to a function F_1 on $\mathcal{U} \times^B G$. Since $\mathcal{U} \times G \rightarrow \mathcal{U} \times^B G$ is a locally trivial fibration, we see that F_1 is a regular function.

Since F_1 goes down to F and $\mathcal{U} \times^B G \longrightarrow \widehat{X}$ is a proper morphism, we see that if Γ is the graph of F , then the canonical morphism $\Gamma \longrightarrow \widehat{X}$ is *proper and bijective*. Then from Lemmas 3.6.5 and 3.6.6 given below, it follows that F^q is a regular function on \widehat{X} for $q = p^r$, $r \gg 0$.

Lemma 3.6.5. (Cf. [K1, pp. 260–261, Lemma 1.4]) *Let $j : P_1 \longrightarrow P_2$ be a proper, bijective morphism of algebraic schemes. Then the following hold:*

- (1) *If f is a regular function on P_1 , then f^{p^r} is the pull-back of a regular function on P_2 for $r \gg 0$.*
- (2) *If g_1 and g_2 are regular functions on P_2 such that $j^*(g_1) = j^*(g_2)$, then $g_1^{p^r} = g_2^{p^r}$ for $r \gg 0$.*

Proof. Since $(g_1 - g_2)^{p^r} = g_1^{p^r} - g_2^{p^r}$, to prove the last assertion, it suffices to show that if $j^*(g) = 0$ for a regular function g on P_2 , then $g^{p^r} = 0$ for $r \gg 0$. The question is local so that we can suppose that $P_2 = \text{Spec } A$, $P_1 = \text{Spec } B$ and j is given by a homomorphism $j^* : A \longrightarrow B$. The hypothesis implies that $I = \ker j^*$ is nilpotent and since $g \in I$, $g^{p^r} = 0$ for $r \gg 0$. This proves the last assertion.

Now the question of proving the first assertion becomes local. To see this let $\{U_i\}$ be an open cover of P_1 and $\{U'_i\}$ the open cover of P_2 such that $U_i = j^{-1}(U'_i)$. Let f_i be the restriction of f to U_i and suppose that there exist g_i in U'_i such that $f_i = j^*(g_i)$. Then we see that for $r \gg 0$, $g_i^{p^r} = g_j^{p^r}$ in $U'_i \cap U'_j$ so that $(g_i)^{p^r}$ define a regular function g on P_2 and we have $f^{p^r} = j^*(g^{p^r})$.

We claim that we can suppose that P_1 is reduced. For, consider $(P_1)_{\text{red}} \xrightarrow{i} P_1 \xrightarrow{j} P_2$, where i is the canonical morphism $(P_1)_{\text{red}} \longrightarrow P_1$. Let $f' = i^*(f)$, g a regular function on P_2 such that $(j \circ i)^*(g) = f'$ and $f_1 = j^*(g)$. Then $i^*(f_1) = i^*(f)$ so that by the above considerations we have $f_1^{p^r} = f^{p^r}$ for $r \gg 0$, which implies that $j^*(g^{p^r}) = f^{p^r}$ for $r \gg 0$. The claim follows.

We claim that we can also suppose that P_2 is reduced. Then we have the factorisation for j ($P_1 = (P_1)_{\text{red}}$)

$$\begin{array}{ccc} P_1 & \xrightarrow{j} & P_2 \\ & \searrow j' & \downarrow i \\ & & (P_2)_{\text{red}} \end{array}$$

Let g' be a regular function on $(P_2)_{\text{red}}$ such that $f = (j')^*(g')$. We see that there is a regular function g on P_2 such that $i^*(g) = g'$, for the considerations being local (as observed above) we can suppose that $P_2 = \text{Spec } A$ so that $(P_2)_{\text{red}} = \text{Spec } A/I$ and the existence of g follows. We have then $j^*(g) = f$ and the claim follows.

We claim that P_1 (and hence P_2) can also be taken to be irreducible. Let $(P_1)_i$ ($1 \leq i \leq r$) be the irreducible components of P_1 and $(P_2)_i = j((P_1)_i)$ the irreducible

components of P_2 . Let f_i be the restriction of f to $(P_1)_i$ and g_i a regular function on $(P_2)_i$ such that $j^*(g_i) = f_i$. Then g_i and g_j coincide set theoretically on $(P_2)_i \cap (P_2)_j$ so that as we saw above in the proof Proposition 3.6.4, $(g_i)^{p^r}$ patch up to define a regular function g on P_2 for $r \gg 0$ and then $f = j^*(g)$, which proves the claim.

Thus we can suppose that P_1 and P_2 are both reduced and irreducible. The considerations being local we can thus suppose that $P_1 = \text{Spec } B$, $P_2 = \text{Spec } A$, where A and B are integral domains and $j^* : A \rightarrow B$ is an inclusion, B being integral over A . If $k(A)$ and $k(B)$ are the quotient fields of A and B respectively, our hypothesis makes $k(B)$ a purely inseparable extension over $k(A)$ of degree $q = p^r$. Then $B^q \subset k(A)$ and elements of B^q are integral over A . Then if $B_1 = A \cdot B^q$, we have $A \hookrightarrow B_1 \hookrightarrow B$, $k(B_1) = k(A)$ and $\text{Spec } B_1 \rightarrow \text{Spec } A$ is proper bijective and to prove the lemma it suffices to consider this case.

Thus to prove the lemma we can also suppose that $k(A) = k(B)$. Then the conductor $C = \{b \in B \mid bB \subset A\}$ is a non-zero ideal in B (as well as A) and the map of schemes

$$\text{Spec } B/C \rightarrow \text{Spec } A/C$$

induced by $A/C \hookrightarrow B/C$ is a proper bijection and $\dim B/C < \dim B$. We now prove the lemma by induction on the dimension of B . Let \bar{f} be the image of $f \in B$ in B/C . Then by the induction hypothesis, there is a $\bar{g} \in A/C$ such that \bar{g} maps to \bar{f}^q , $q = p^r$, $r \gg 0$. Let g be a representative of \bar{g} in A . Then we see that $g + \theta = f^q$ for $\theta \in C$ and this proves the lemma. \square

Recall that a line bundle L on a projective scheme P is said to be *semi-ample* if given $x \in P$, there is a section of a power of M which does not vanish at x .

Lemma 3.6.6. *Let $j : P_1 \rightarrow P_2$ be a proper, bijective morphism of algebraic schemes, L a line bundle on P_2 and M the line bundle $j^*(L)$ on P_1 . Then given a section 's' of M , s^q is the pull-back of a section of L^q , $q = p^r$, $r \gg 0$. In particular, we have*

$$M \text{ semi-ample} \iff L \text{ semi-ample}.$$

Proof. The implication \Leftarrow is trivial. The reverse implication is an easy consequence of Lemma 3.6.5. In greater detail, let θ_{ij} be the transition functions of L with respect to an covering $\{U_i\}$ of P_2 and $\{U'_i\}$ the open covering of P_1 given by $U'_i = j^{-1}(U_i)$. Then $\theta'_{ij} = j^*(\theta_{ij})$ are transition functions of M with respect to the covering $\{U'_i\}$. The section 's' of M is given by regular functions s_i in U'_i such that

$$s_j = s_i \theta'_{ij} \text{ in } U'_i \cap U'_j.$$

Then by the above lemma, s_i^q is the pull-back of a regular function t_i in U_i , $q = p^r$, $r \gg 0$. Then the pull-backs of t_j and $t_i \theta_{ij}^q$ coincide on $U_i \cap U_j$, since $s_j^q = s_i^q \theta_{ij}^q$. Thus again by applying Lemma 3.6.5 (by taking a suitable q th power), we can

indeed suppose that $t_j = t_i \theta_{ij}^q$ in $U_i \cap U_j$ i.e. $\{t_i\}$ patch upto define a section ‘ t ’ of L^q and we see that s^q is the pull-back of the section ‘ t ’. \square

It remains to prove the Proposition 3.6.4 for the case X_1 (or \widehat{X}_1). First observe that if we have a (regular) section s of L on X_1 , then a suitable power of s extends to the cone \widehat{X}_1 . Thus it suffices to prove the slightly weaker version that there exists a G -invariant section s on X_1 of a suitable power of L such that $s(x) \neq 0 \forall x \in X_1^{ss}$. Since X_1^{ss} is a single orbit, X_1^{ss} is smooth. Let $p : X_2 \rightarrow X_1$ be the normalisation of X_1 . We have a canonical G -action on X_2 and p is a G -morphism. Then the pull-back $p^*(L)$ of L is ample on X_2 and if X_2^{ss} denotes the semi-stable points of X_2 for $p^*(L)$, one knows that

$$(iv) \quad x \in X_2^{ss} \iff p(x) \in X_1^{ss}.$$

Now $p : X_2^{ss} \rightarrow X_1^{ss}$ is an isomorphism and X_2^{ss} is a G -orbit. Suppose that there is a G -invariant section of $p^*(L)$ (or a suitable power) such that $s(x) \neq 0 \forall x \in X_2^{ss}$. Then s vanishes on $X_2 \setminus X_2^{ss}$ and if J denotes the ideal sheaf on X_2 obtained as the ideal sheaf on $X_1 \setminus X_1^{ss}$ on X_1 , then a suitable power of s belongs to J (the support of J is $X_2 \setminus X_2^{ss}$). Then we see that this power of s comes from a section of a power of L on X_1 , having the required properties.

Thus without loss of generality, we can suppose that X_1 is *normal*.

Since X_1^{ss} is a G -orbit, $X_1^{ss} \simeq H \setminus G$, the restriction of L^n to X_1^{ss} is trivial ($n \gg 0$). Hence without loss of generality we can suppose that there is a regular G -invariant section of L on X_1^{ss} . We shall now show that s (or a suitable power of s) extends to a regular section on X_1 . This assertion is true if G happens to be a torus, in particular \mathbb{G}_m . (For given $x \in X_1^{ss}$, we can find a G -invariant regular section θ of L^r (for some r) such that $\theta(x) \neq 0$. Then s^r and a constant multiple of θ coincide on X_1^{ss} and hence they coincide everywhere). Suppose that s is not regular on X_1 . Then s has a *pole* at some $x_0 \in X_1 \setminus X_1^{ss}$ (i.e. s is not regular at x_0 and has no indeterminacy at x_0). Choose some $x \in X_1^{ss}$. Then there is a K -valued point $g \in G(K)$ (or possibly by going to a finite extension of K , K being the quotient field of the d.v.r. A as in Remark 3.5.1) such that $\lim_{t \rightarrow 0} x \circ g = x_0$. Now by the Iwahori-Matsumoto theorem, we have $g = P\lambda Q$, where $P, Q \in G(A)$ and $\lambda : \mathbb{G}_m \rightarrow G$ is a non-trivial 1-PS, so that the image of λ is a subgroup H of G , $H \simeq \mathbb{G}_m$ and λ defines a K -valued point of G . We see that $\lim_{t \rightarrow 0} x \circ P\lambda = x'_0 \in X_1 \setminus X_1^{ss}$ and s has a pole at x'_0 . Let $P_0 = \lim_{t \rightarrow 0} P$. Then $P_0 \in G$ and $x \cdot P_0 \in X_1^{ss}$. We have $x \circ P\lambda = x \cdot P_0 \cdot P_0^{-1} P\lambda$ and $\lim_{t \rightarrow 0} P_0^{-1} P = I$ (identity element of G). Thus without loss of generality, we can suppose that $\lim_{t \rightarrow 0} x \circ \lambda = x_0 \in X_1 \setminus X_1^{ss}$ and s has a pole at x_0 . Let Z denote the closure of the “ λ -orbit through x ” i.e. the closure of xH in X_1 and take the restriction θ of s to Z . Then θ is H -invariant and has a

pole at x_0 (which is in Z) and $Z^{ss} = xH$. This reduces the problem to $G = \mathbb{G}_m$, where as we observed above, the assertion is true i.e. a power of θ is regular on X_1 . This leads to a contradiction of the hypothesis that s has a pole at x_0 . This proves Proposition 3.6.4. \square

Definition 3.6.7. Let $A = k[[t]]$ and let K be its quotient field. We say “ X^{ss} mod G is proper” if given $x \in X^{ss}(K)$ there exists a finite extension K' of K and $g \in G(K')$ such that $x \circ g \in X^{ss}(A')$ where A' is the finite extension of A associated with K' .

Corollary 3.6.8. [S2, p. 526, Thm. 4.1] *Let X be a closed G -invariant subscheme of $\mathbb{P}(V)$. Then X^{ss} mod G is proper.*

Proof. As in Remark 3.5.1, let Z_K denote the closure of the G_K orbit $Z_K^0 = G_K$ in X_K and Z_A the flat closure of Z_K in X_A . Now x is in Z_K^{ss} (in the sense of Subsection 3.5). Let \overline{K} denote the algebraic closure of K , then taking $Z_{\overline{K}}$, $X_{\overline{K}}$ etc. we see that $Z_{\overline{K}}^0$ is an orbit under $G_{\overline{K}}$, $Z_{\overline{K}}$ is the closure of $Z_{\overline{K}}^0$ and $Z_{\overline{K}}^0 \subset Z_{\overline{K}}^{ss}$. Then by Proposition 3.6.4, there exists a (regular) section s of $L_{\overline{K}}$ (or a suitable power of $L_{\overline{K}}$) on $Z_{\overline{K}}$ such that s is $G_{\overline{K}}$ invariant and s is non-vanishing at every point of $Z_{\overline{K}}^{ss}$. Now s is defined over a finite extension of K and thus without loss of generality we can suppose that there exists a section s of L_K over Z_K which is G_K invariant and does not vanish at every closed or geometric point of Z_K^{ss} .

We claim that to prove the corollary, it suffices to prove that Z_k^{ss} is not empty, where Z_k denotes the closed fibre of $Z_A \rightarrow \text{Spec } A$ (we see that this is also necessary). Suppose that $Z_k^{ss} \neq \emptyset$, then we see easily that there exists $y \in Z_A(A)$ (or possibly we may have to go a finite extension of K , take the integral closure of A in this extension etc.) such that its restriction to the closed fibre is y_0 and restriction to the generic fibre is in Z_K^0 (we see that $y \in Z_A^{ss}(A) \subset X_A^{ss}(A)$). We now see that there exists $g \in G(K)$ such that $x \circ g = y$ (again we may have to go a finite extension of K for this assertion).

Let $p : \tilde{Z}_A \rightarrow Z_A$ be the normalisation of Z_A . We take semi-stable points in \tilde{Z}_A with respect to the pull-back of L_A by p . It suffices to prove that $\tilde{Z}_k^{ss} \neq \emptyset$, where \tilde{Z}_k denotes the closed fibre of $\tilde{Z}_A \rightarrow \text{Spec } A$, for $\tilde{Z}_k^{ss} = p^{-1}(Z_k^{ss})$. Let \tilde{s} denote the pull-back of the section s of L_K on Z_K . Then by multiplying \tilde{s} by a suitable power of the uniformising parameter π of A , we can suppose that \tilde{s} is a regular section of L_A on \tilde{Z}_A . The restriction \tilde{s}_k of \tilde{s} to \tilde{Z}_k is obviously $G(= G_k)$ invariant. Then if \tilde{s}_k does not vanish at $z \in \tilde{Z}_k$, then obviously $z \in \tilde{Z}_k^{ss}$. Thus it suffices to prove that \tilde{s}_k is not identically zero on \tilde{Z}_k . We claim that there exist q, r such that $q \geq 0$ and $r \geq 1$ such that $\pi^{-q}s^r$ is regular on \tilde{Z}_A and its restriction to \tilde{Z}_k is not identically zero. This would suffice. To prove this claim let m_i (resp. n_i) denote the order of vanishing of π (resp. \tilde{s}) along the irreducible components Z_k^i of

\tilde{Z}_k (\tilde{Z}_k is a divisor of the normal scheme \tilde{Z}_A). Then the order of $\pi^{-q}s^r$ along Z_k^i is $(rn_i - qm_i)$ (note that $m_i \geq 1$ and $n_i \geq 0 \forall i$). To prove the claim we have only to show that $(rn_i - qm_i) \geq 0 \forall i$ and $\exists i_0$ such that $rn_{i_0} - qm_{i_0} = 0$. If we take i_0 such that $\frac{n_{i_0}}{m_{i_0}} = \min_i \frac{n_i}{m_i}$ and q, r such that $\frac{q}{r} = \frac{n_{i_0}}{m_{i_0}}$, we are done. This proves the corollary. \square

Remark 3.6.9. We see that in the proof of Proposition 3.6.4, the argument actually shows that if \hat{X} and \hat{X}_1 are closed G -invariant subsets of V such that $\hat{X} = \mathcal{U}(\hat{X}_1)$ and f is a G -invariant function on \hat{X}_1 , then f extends (uniquely) to a G -invariant function on \hat{X} (and similar assertion for the case X, X_1). Again there is a more general assertion concerning the extension of a G -invariant section of L on X^{ss} to the whole of X (see [S2, p. 526, Theorem 4.1]).

4. Stratified spaces

Throughout this section we fix a *reduced and irreducible* quotient data

$$(4.0.10) \quad \Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y).$$

This section consists largely of definitions which facilitate the proof of the main theorem.

4.1. Stratification of $(X^{ss} // G)_{\text{top}}$.

Lemma 4.1.1. *There exists a non-empty open subset $U \subset Y$ such that*

- (i) *U has the structure of a k -variety;*
- (ii) *The map $\alpha_{X,U}: \alpha_X^{-1}(U) \rightarrow U$ is a map of k -schemes;*
- (iii) *The structure sheaf on U is given by the G -invariant direct image of $\mathcal{O}_{\alpha_X^{-1}(U)}$, i.e., by the sheaf $V \mapsto \Gamma(\alpha_X^{-1}(V), \mathcal{O}_X)^G$ (V open in U).*

Moreover, a power of the line bundle L on X^{ss} descends to a line bundle on U and by shrinking U , this descended line bundle may be assumed to be trivial.

Proof. Let X be a closed G -invariant subset of $\mathbb{P}(V)$ and $\alpha: X^{ss} \rightarrow Y$ (or α_X) the map as above. Let us suppose that Y is *irreducible*. Let e be the maximum of the dimension of the closed orbits in X^{ss} . Recall (Corollary 3.4.7 to Proposition 3.4.1) that the subset $X^{ps,e}$ of closed orbits of dimension e is open in $X^{ss,e}$, the closed subset of orbits of dimension $\leq e$ in X^{ss} (we endow X, X^{ss} etc. with the reduced scheme structures). The canonical map $X^{ss,e} \rightarrow Y$ is surjective and $X^{ss,e} \setminus X^{ps,e}$ maps onto a closed subset of Y which is not the whole of Y so that $X^{ps,e}$ maps onto a non-empty open subset of Y . Then by the irreducibility of Y , it is seen without much difficulty (noting that G is irreducible) that there exists a non-empty *open irreducible G -invariant* subset of $X^{ps,e}$ which maps onto an open subset ($\neq \emptyset$) of Y and we denote the closure of this subset in X by X_0 . Thus we have a *closed*,

irreducible G -invariant subset X_0 of X such that the canonical map $\alpha (= \alpha_{X_0}) : X_0^{ss} \rightarrow Y$ is surjective and we have a non-empty, G -invariant open subset of $X_0^{ps,e}$ as well as of X_0^{ss} , consisting of the orbits of dimension e , which are closed in X_0^{ss} and hence in X^{ss} (polystable orbits of dimension e , $X_0^{ps,e} = X_0^{ps,e} \cap X_0^{ss}$). By the existence of *generic quotients*, we see easily that there is a G -stable open subset of X_0^{ss} , which is therefore of the form $\alpha_{X_0}^{-1}(U)$, where U is open in Y such that the geometric quotient $\alpha_{X_0}^{-1}(U) \text{ mod } G$ exists and is a variety (for example by arguments in [K1] or [S4]). Since $\alpha_{X_0}^{-1}(U) \text{ mod } G$ identifies with U (set-theoretically and topologically) we can thus endow U with the structure of a variety. We denote this variety structure on U by U_0 so that the structure sheaf on U_0 is given by the sheaf of G -invariant regular functions on $\alpha_{X_0}^{-1}(U)$. Observe that the generic fibre of $\alpha_{X_0}^{-1}(U) \rightarrow U$ is of the form “ $G \text{ mod } H$ ”, (H the isotropy group at a polystable point) and therefore by suitably shrinking U , we can suppose that there exists a G -invariant regular section of a (suitable power of L) on $\alpha_{X_0}^{-1}(U)$, which is everywhere non-vanishing (see Proposition 3.6.4 and Subsection 3.5 above), in particular, (a suitable power of) L on $\alpha_{X_0}^{-1}(U)$ descends to U . Consider the map $\alpha_X : X^{ss} \rightarrow Y$ and the map $\alpha_X : \alpha_X^{-1}(U) \rightarrow U$ induced by it. We claim that given a G -invariant regular function on $\alpha_{X_0}^{-1}(U)$, a suitable p th power of this function extends to a regular (G -invariant) function on $\alpha_X^{-1}(U)$. This follows by an extension of the arguments in Proposition 3.4.1 and Proposition 3.6.4, since we have

$$\mathcal{U}(\alpha_{X_0}^{-1}(U)) = \alpha_X^{-1}(U)$$

where we define the left hand side by

$$\mathcal{U}(\alpha_{X_0}^{-1}(U)) = X^{ss} \setminus (X_0^{ss} \setminus \alpha_{X_0}^{-1}(U)).$$

Thus we have a variety structure on U defined by G -invariant regular functions on $\alpha_X^{-1}(U)$, which we denote simply by U . Then we have a morphism $U_0 \rightarrow U$ which is proper and bijective. Now the generic fibre of $\alpha_X^{-1}(U) \rightarrow U$ is a single semi-stable equivalence class and by Proposition 3.6.4, by suitably shrinking U , we see that there is a regular G -invariant section of L on $\alpha_X^{-1}(U)$, which is everywhere non-vanishing. In particular, we see that L on $\alpha_X^{-1}(U)$ goes down to a line bundle on U (we can assume it to be trivial). \square

Definition 4.1.2. We define the dimension of Y to be $\dim Y = \dim U$, where U is the non-empty open set of Lemma 4.1.1. The function field of Y , $k(Y)$ is defined by the formula $k(Y) = k(U)$. Note that these notions are well-defined.

Remark 4.1.3. Note that Y possesses a stratification $\{U_\lambda\}$ by locally closed subsets $U_\lambda \subset Y$, each of which have the structure of a k -variety. Indeed, setting $Y' = Y \setminus U$, and $X' = \alpha_X^{-1}(Y')$, we note that $Y' = ((X')^{ss}(L) // G)_{\text{top}}$. Since $\dim Y' < \dim Y$,

by induction we see the stratification of Y . In particular, by repeatedly using the last statement of Lemma 4.1.1, we see that there is a positive integer n such that $L^n|_{\alpha_X^{-1}(y)}$ is trivial on every fibre $\alpha_X^{-1}(y)$ of α_X .

Definition 4.1.4. Let $\{U_\lambda\}$ be the stratification in Remark 4.1.3. Let W be a k -scheme, together with locally closed subschemes $\{W_\lambda\}$ which give a stratification of W (i.e., the W_λ are disjoint and as sets their union equals W). A continuous map of $q: W \rightarrow Y$ is said to be a *stratified morphism* or a *stratified map* if, for each index λ , $W_\lambda = q^{-1}(U_\lambda)$ and the map

$$W_\lambda = q^{-1}(U_\lambda) \xrightarrow{q|_{W_\lambda}} U_\lambda$$

is a map of k -schemes.

Definition 4.1.5. Let $\{U_\lambda\}$ a stratification of Y as in Remark 4.1.3, and W a k -variety. A stratified map $q: W \rightarrow Y$ is said to be *generically finite* if W is *complete* and

$$q^{-1}(U) \xrightarrow{\text{via } q} U$$

is a generically finite map of k -varieties.

Note that if $q: W \rightarrow Y$ is generically finite, then the field extension $k(Y) \rightarrow k(W)$ is finite.

Lemma 4.1.6. *Let W be a normal projective variety and $q: W \rightarrow Y$ be a generically finite map. Then there exists a normal projective variety W' such that*

- (i) *There is a surjective map of varieties $W' \rightarrow W$ such that the composite*

$$W' \rightarrow W \xrightarrow{q} Y$$

is generically finite. We denote this composite q' .

- (ii) *The field $L := k(W')$ is the least normal extension of $k(Y)$ which contains $k(W)$.*
- (iii) *If $\Gamma := \text{Aut}_{k(Y)}(L)$, then Γ acts on W' and the orbits of W' under Γ are contained in the fibres of q' . In other words, as a continuous map, q' is Γ -invariant for the trivial action of Γ on Y .*
- (iv) *There exists a non-empty open subset V of the “big stratum” U of Y such that for $v \in V$ the fibre $(q')^{-1}(v)$ is an orbit of Γ .*

Proof. Set $n = [k(W) : k(Y)]_s$, the separable degree of $k(W)/k(Y)$. Let $R = k(W)^{\otimes n}$, the tensor product being taken over $k(Y)$. Then L can be expressed as a quotient of the $k(Y)$ -algebra R . Let $I = \ker(R \rightarrow L)$. The symmetric group S_n on n -letters acts on R in an obvious way, the group Γ can be identified with the subgroup of S_n which leaves I invariant.

The n -fold fibre-product $W_n := W \times_Y \cdots \times_Y W$ makes sense as a closed subscheme of the n -fold product $W \times_k \cdots \times_k W$. Now, S_n also acts on W_n and one can find a closed subvariety W' of W_n , such that, if one has $k(W') = R/I = L$. Through projection into one of the factors we get a surjective morphism $W' \rightarrow W$ of projective varieties which at the generic points is represented by the function field extension $K(W) \rightarrow L$. Let The group Γ acts on W' and this action lifts to the normalisation of W' . Replacing W' by this normalisation, and composing $W' \rightarrow W$ with q we get a generically finite map

$$q': W' \rightarrow Y.$$

We claim that if $x, y \in W'$ with $y = \gamma x$, $\gamma \in \Gamma$ then $q'(x) = q'(y)$. This is certainly true over U . Thus q' and $q' \circ \gamma$ agree over $(q')^{-1}(U)$ which is a Γ -stable set. Hence q' and $q' \circ \gamma$ agree on all of W' . This proves (i)—(iii) and assertion (iv) is clear. \square

The notion of a stratified map seems to depend on the chosen stratification of Y . However there are maps $q: W \rightarrow Y$ which are stratified, whatever be the stratification of Y into a finite number of locally closed subsets U_λ with a k -variety structure such that \mathcal{O}_{U_λ} is the sheaf of G -invariant sections on $\alpha_X^{-1}(U_\lambda)$. This happens when W carries a principal G -bundle $Z \rightarrow W$ and Z has an equivariant map into X^{ss} . For this it is perhaps best to state matters in terms of stacks.

4.2. Stack theoretic interpretation. Recall that in this section our fixed quotient data (4.0.10) is reduced and irreducible. There is a distinguished index $\lambda = \circ$ such that U_\circ is an open subset of Y . Set $X_\lambda := \alpha_X^{-1}(U_\lambda)$ and $\alpha_\lambda: X_\lambda \rightarrow U_\lambda$ equal to the restriction of α_X to (the locally closed) subvariety X_λ of X^{ss} . Since the structure sheaf \mathcal{O}_{U_λ} is given by the sheaf of G -invariant sections of X_λ , we see that $\alpha_X: X^{ss} \rightarrow Y$ is a stratified map, whatever be the stratification on Y induced by Lemma 4.1.1 (i.e., by the choices of the open subsets predicted by Lemma 4.1.1).

Recall that the stack $[X^{ss}/G]$ is the fibered category over k -schemes such that for any k -scheme Q , $[X^{ss}/G](Q)$ is the category whose objects are pairs $(P \xrightarrow{\delta} Q, P \xrightarrow{j} X^{ss})$ with $\delta: P \rightarrow Q$ a principal G -bundle and $j: P \rightarrow X^{ss}$ a G -equivariant map. Morphisms in $[X^{ss}/G](Q)$ are isomorphisms of such pairs. $[X^{ss}/G]$ is a stack with respect to either the fppf or the étale topologies on the category of k -varieties. One could restrict to (and we will do so) base k -schemes Q which are k -varieties.

Given the data $(Y, \{U_\lambda\})$, one can define another stack (also denoted Y) and a map of stacks $\gamma: [X^{ss}/G] \rightarrow Y$ as follows. For a k -variety W , let

$$(4.2.1) \quad \underline{Y}(W) = \{q: W \rightarrow Y \mid q \text{ is a stratified map}\}.$$

Let $W' \rightarrow W$ be a faithfully flat and finitely presented map of varieties, $W'' := W' \times_W W'$ the two-fold product of W' with itself over W , and $p_i: W'' \rightarrow W'$, $i = 1, 2$ the two projections. It is easy to see that if we have a stratified map

$q': W' \rightarrow Y$ such that $q' \circ p_1 = q' \circ p_2$, then there is a unique stratified map $q: W \rightarrow Y$ such that q' is the composite $W' \rightarrow W \xrightarrow{q} Y$. In fact descent works on each stratum of W' , giving us, set-theoretically a map $q: W \rightarrow Y$, which is a map of varieties on each $q^{-1}(U_\lambda)$. Now $q': W' \rightarrow Y$ is continuous and $W' \rightarrow W$ being fppf, W has the quotient topology from W' . It follows that q is continuous. Thus \underline{Y} is also a stack — since it is a sheaf of sets on the fppf site over k -varieties.

Given a principal G -bundle $\delta: P \rightarrow Q$ and a G -equivariant map $j: P \rightarrow X^{ss}$, we have, clearly a continuous map

$$(4.2.2) \quad q = q(\delta, j): Q \rightarrow Y.$$

We point out that $q(\delta, j)$ does not depend on the stratification $\{U_\lambda\}$ that of Y that we've fixed. We now proceed to show that q is stratified (and hence is stratified for every stratification as in Remark 4.1.3). If the locally closed subset $Q_\lambda = q^{-1}(U_\lambda)$ of Q is given the reduced topology, and $P_\lambda = \delta^{-1}(Q_\lambda)$, then we have a G -equivariant map of varieties $j_\lambda: P_\lambda \rightarrow X_\lambda$. The map $\delta_\lambda: P_\lambda \rightarrow Q_\lambda$ (given by restricting δ to P_λ) is a principal bundle. It follows (from the universal properties of the geometric quotient Q_λ of P_λ by G) that we have a map of varieties $q_\lambda: Q_\lambda \rightarrow Y$. Moreover, clearly, $q_\lambda = \gamma|_{Q_\lambda}$. Thus $q(\delta, j)$ is a stratified map. The maps $q(\delta, j)$ induce a map of stacks

$$(4.2.3) \quad \gamma: [X^{ss}/G] \rightarrow \underline{Y}.$$

such that if $f: Q \rightarrow [X^{ss}/G]$ is the classifying map for the data (δ, j) then $\gamma \circ f$ is the map whose underlying stratified map is $q(\delta, j)$.

From now on, we will identify the stack \underline{Y} with Y . The main theorem (when proved) will show that Y is actually a variety. But for the moment, it is not even clear it is an algebraic stack (i.e., that it has a presentation by a scheme which is smooth over it).

Note that $[X^{ss}/G]$ can morally be regarded as a “stratified space”, the “stratification” being given by $\{[X_\lambda/G]\}$. Note also that morally $\gamma: [X^{ss}/G] \rightarrow Y$ can be regarded a “stratified map”.

Convention 4.2.4. We observed in Remark 4.1.3 that there exists a positive integer n such that L^n is trivial on the fibres of α_X . Since $X^{ss}(L) = X^{ss}(L^n)$, and $X^s(L) = X^s(L^n)$, we may assume, by replacing L by L^n if necessary, that L is trivial on the fibres of α_X . For the rest of our discussion we make this assumption.

In view of the above convention, if $(P \rightarrow Q, P \rightarrow X^{ss})$ is an object in $[X^{ss}/G](Q)$, then the pull back of L to P is trivial on the fibres of $P \rightarrow Q$ and hence descends to a line bundle on Q . In other words $L|_{X^{ss}}$ gives us a line bundle on $[X^{ss}/G]$. We make the following definition:

Definition 4.2.5. The line bundle on $[X^{ss}/G]$ induced by L on X^{ss} via descent will be denoted $L_{/G}$.

Now suppose we have a Q -valued point of $[X^{ss}/G]$ where Q is a k -scheme. In other words suppose we have a principal G -bundle $\delta: P \rightarrow Q$ as well as a G -equivariant map $j: P \rightarrow X^{ss}$. The line bundle j^*L descends to a line bundle L_Q on Q . In fact the line bundle L_Q can be regarded as $f^*L_{/G}$ where $f: Q \rightarrow [X^{ss}/G]$ is the natural map induced by the data (δ, j) . There is also a resulting stratified map $i: Q \rightarrow (X^{ss} // G)_{\text{top}}$. In fact $i = \gamma \circ f$. It turns out that L_Q depends only on i rather than on (δ, j) , if Q is a normal variety. In other words, if Q is normal and the base of another principal G -bundle with an equivariant map to X^{ss} such that the resulting stratified map $Q \rightarrow (X^{ss} // G)_{\text{top}}$ is again i , then, the line bundle induced on Q from this principal bundle is again L_Q . One can say it better in the following way:

Proposition 4.2.6. *Let (X, L) be a G -pair and suppose $Y = (X^{ss} // G)_{\text{top}}$ is irreducible. Suppose Q is a normal k -variety and we have two maps $Q \xrightarrow{f} [X^{ss}/G]$ and $Q \xrightarrow{g} [X^{ss}/G]$ such that $\gamma \circ f = \gamma \circ g$ as stratified maps from Q to Y , i.e., suppose the diagram*

$$\begin{array}{ccc} Q & \xrightarrow{f} & [X^{ss}/G] \\ g \downarrow & & \downarrow \gamma \\ [X^{ss}/G] & \xrightarrow{\gamma} & Y \end{array}$$

commutes. Then, with $L_{/G}$ as in 4.2.5 (see also 4.2.4), we have

$$f^*L_{/G} \simeq g^*L_{/G}.$$

Proof. Note that we have a canonical topology on the product space $Q \times Y$. The closed subsets in $Q \times Y$ are the images in $Q \times Y$ of the closed G -invariants subsets of $Q \times X^{ss}$ (for the trivial action of G on Q), under the canonical map $Q \times X^{ss} \rightarrow Q \times Y$ (or equivalently closed subsets of $Q \times X^{ss}$ which are saturated for this map). The closed subsets of $Q \times Y$ can also be defined as follows. Observe that $Q \times Y$ has also a stratification by subspaces which are schemes (we take the product of Q with the subschemes defining a stratification of Y). We say that a subset C of $Q \times Y$ is closed if the intersection C_i of C with every subscheme in this stratification is closed and satisfies a property for limits expressed by a valuation criterion as follows. Let A be a d.v.r. and K its quotient field. An A -valued point θ of $Q \times Y$ (written $\text{Spec } A \rightarrow Q \times Y$) is one which can be “lifted” to an A -valued point of $Q \times X^{ss}$.

$$\begin{array}{ccc}
\text{Spec } A & \longrightarrow & Q \times_k X^{ss} \\
& \searrow \theta & \downarrow 1 \times \alpha \\
& & Q \times Y
\end{array}$$

We see that θ can be viewed as a set theoretic map of a neighbourhood of the closed point of the smooth curve defined by $\text{Spec } A$, into $Q \times Y$. Suppose that θ defines a K -valued point of some C_i . Then the condition to be imposed is that θ maps the closed point of $\text{Spec } A$ to a point of C .

Let $f: Q \rightarrow [X^{ss}/G]$ be given by the data $(P \xrightarrow{\delta} Q, P \xrightarrow{j} X^{ss})$. Let $i: Q \rightarrow Y$ be the resulting map (i.e. $i = \gamma \circ f$). We have a commutative diagram:

$$\begin{array}{ccc}
P & \xrightarrow{j} & X^{ss} \\
\delta \downarrow & & \downarrow \alpha \\
Q & \xrightarrow{i} & Y
\end{array}$$

The map i is “nice” in the following sense. In fact, it is not difficult to see that if Γ_0 is the graph of j and we take the canonical action of $G \times G$ on $P \times X^{ss}$ and take the closure Γ of Γ_0 . $(G \times G)$ in $P \times X^{ss}$, then the canonical image of Γ in $Q \times Y$ is the graph of i . This can be expressed more intuitively as follows. Let θ be a K -valued point of $Q \times Y$, which is the image of a K -valued point $\phi = (\phi_1, \phi_2)$ of the graph of j , ϕ_1 being a K -valued point of P and ϕ_2 a K -valued point of X^{ss} . Since $P \bmod G$ and $X^{ss} \bmod G$ are proper, we see that there exist K -valued points g_1, g_2 of G such that $\psi_1 = \phi_1 \circ g_1$ (resp. $\psi_2 = \phi_2 \circ g_2$) is an A -valued point of P (resp. X^{ss}). We see that $j(\psi_1) = \psi_2 \circ g$, where $g = g_2^{-1}g_1$ is a K -valued point of G (using the fact that $j(\phi_1) = \phi_2$ and j is G -equivariant). Now $j(\psi_1)$ is an A -valued point of X^{ss} so that the closed points of X^{ss} determined by $j(\psi_1)$ and ψ_2 are semi-stably equivalent i.e. they are in the same fibre of $X^{ss} \rightarrow Y$ and determine a point y of Y . Let x be closed point of Q determined by ψ . Then we see that (x, y) is in the graph of i and this essentially shows that the graph of i is closed.

Consider the “base change” $X' = Q \times_Y X^{ss}$. Then X' is a closed G -stable subset of $Q \times_Y X^{ss}$ (since $Q \times X^{ss} \rightarrow Y \times Y$ is continuous and the diagonal is closed in $Y \times Y$). We endow X' with the canonical reduced structure as a closed subscheme of $Q \times X^{ss}$. We have a canonical morphism j' of P into X' . In fact we have a

commutative diagram (with the parallelogram being “cartesian”):

$$(4.2.7) \quad \begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ P & \xrightarrow{j'} & X' & \xrightarrow{j''} & X^{ss} \\ & \searrow \delta & \swarrow \alpha' & & \swarrow \alpha \\ & & Q & \xrightarrow{i} & Y \end{array}$$

Our strategy for establishing the Proposition is to prove that for a non-empty set V of Q we have a natural isomorphism—behaving well with respect to restrictions to open subschemes of V :

$$(4.2.8) \quad \Gamma(V, f^*L/G) \xrightarrow{\sim} \Gamma((\alpha')^{-1}(V), (j'')^*L).$$

The right side depends only on the map $i: Q \rightarrow Y$ and not on f (for the space X' depends only on i), whence establishing (4.2.8) is equivalent to establishing the Proposition.

We denote by \overline{X} the closure of $j'(P)$ in X' and endow it with the reduced subscheme structure. Then we have a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\overline{j}} & \overline{X} \\ & \searrow \delta & \swarrow \overline{\alpha} \\ & & Q \end{array}$$

where all the maps are morphisms and \overline{j} is G -equivariant. Observe that if $x \in Q$, $y = i(x)$ and \overline{X}_x denotes the fibre of $\overline{\alpha}$ over x , then \overline{X}_x can be identified with a closed G -stable subset of X_y^{ss} . We denote by the same $L_{\overline{X}}$ the line bundle on \overline{X} , obtained as the pull-back of L on X^{ss} by the canonical morphism $\overline{X} \rightarrow X^{ss}$. Then $\overline{j}^*L_{\overline{X}} = j^*L$.

We can assume without loss of generality that $i(Q) = Y$. Hence $V_0 = i^{-1}(U)$ is a non-empty open subset of Q , where U is the open subset of Y as in Lemma 4.1.1. By *loc.cit.* a suitable power of L descends to U , and this descended bundle can be assumed to be trivial. Without loss of generality, we assume that L itself descends to the trivial bundle \mathcal{O}_U on U . Hence the restriction of $L_{\overline{X}}$ to $(\overline{\alpha})^{-1}(V_0)$ (as well as $j^*L|_{\delta^{-1}(V_0)}$) can be assumed to be trivial. We see, in particular, that the G -invariant sections of j^*L on $\delta^{-1}(V_0)$ identify with the G -invariant sections of $L_{\overline{X}}$ on $(\overline{\alpha})^{-1}(V_0)$. In other words, we have a canonical isomorphism:

$$(4.2.9) \quad \Gamma(V_0, f^*L/G) = \Gamma(\delta^{-1}(V_0), j^*L)^G \xrightarrow{\sim} \Gamma((\overline{\alpha})^{-1}(V_0), L_{\overline{X}})^G.$$

We point out that for any open set V of Q , by definition of f^*L/G we have $\Gamma(V, f^*L/G) = \Gamma(\delta^{-1}(V), j^*L)^G$. For an open subscheme W of \overline{X} , let W_n denote its normalisation, and let $\varphi: \overline{X} \rightarrow X$ denote the normalisation map. We claim that for every non-empty open set V of Q the following holds:

$$(4.2.10) \quad \Gamma(V, f^*L/G) \xrightarrow{\sim} \Gamma([\overline{(\alpha)}^{-1}(V)]_n, \varphi^*L_{\overline{X}})^G$$

In other words, the claim is that the sheaf of sections of f^*L/G on Q identifies with the sheaf of G -invariant regular sections on the normalisation of \overline{X} . This is to be thought of as the first step towards proving (4.2.8), which as we have noted, is sufficient for the Proposition.

Since a G -invariant section of \overline{j}^*L on $(\overline{\alpha})^{-1}(V)$ maps by j^* to a G -invariant section of j^*L on $\delta^{-1}(V)$, to prove (4.2.10) we have only to show that if ‘ s ’ is a section of L on V , then it comes from a G -invariant section of L on the normalisation of $(\overline{\alpha})^{-1}(V)$. Now the restriction of ‘ s ’ to $(V \cap V_0)$ identifies with a G -invariant section of L on $(\overline{\alpha})^{-1}(V \cap V_0)$, so that we can consider ‘ s ’ to be a G -invariant *rational* section of L on $(\overline{\alpha})^{-1}(V)$. Suppose that it is *not* regular on the normalisation of $(\overline{\alpha})^{-1}(V)$. Then we have a non-empty *polar divisor* D for ‘ s ’ so that there is an $x_1 \in D$ such that the “value” of s at x_1 is ∞ (i.e. $x_1 \in D$ and is not a point of indeterminacy). Let $y = \overline{\alpha}(x_1)$. We shall now prove the following:

$$(4.2.11) \quad \overline{X}_y \subset D. \text{ In fact if } z \in \overline{X}_y \text{ the value of ‘} s \text{’ at } z \text{ is } \infty.$$

Now (4.2.11) \implies (4.2.10), for if P_y is the fibre of P over y , $P_y \subset \overline{X}_y$, and ‘ s ’ considered as a section of L over $\delta^{-1}(V)$ would have a polar divisor containing P_y , so that we have a contradiction since ‘ s ’ has been supposed to be regular over $\delta^{-1}(V)$. Thus ‘ s ’ is regular on the normalisation of $(\overline{\alpha})^{-1}(V)$.

To prove (4.2.11), let us first observe that the generic fibre of $\overline{\alpha}: \overline{X} \rightarrow Q$ has a dense G -orbit, since \overline{X} has been defined as the closure of the image of P in $Q \times_Y X^{ss} = X'$. Suppose now $x_2 \in \overline{X}_y$ and the “value” of s at x_2 is *not* ∞ . Hence either ‘ s ’ is regular at x_2 or it is a point of indeterminacy. We then see easily that there is an A -valued point θ_2 (A a.d.v.r. as usual) of \overline{X} such that the closed point of $\text{Spec } A$ maps to x_2 , the K -valued point (K -the quotient field of A) of \overline{X} determined by θ_2 is in the dense G -orbit of the generic fibre of $\overline{\alpha}$ and the “restriction of s to θ_2 ” is regular, i.e. if we set $s(x_2) = \lim_{t \rightarrow 0} s|_{\theta_2}$, then $s(x_2) \neq \infty$. Now we can find another A -valued point θ_1 of \overline{X} such that the closed point maps to x_1 and the K -valued point defined by θ_1 is in the dense G -orbit defined by θ_1 . We see that there is a K -valued point g of G such that $\theta_1 \cdot g = \theta_2$ (we may have to go to a finite extension of K). Since ‘ s ’ is G -invariant and $s(x_1) = \infty$, it follows that $s(x_2) = \infty$, which is a contradiction and the assertion (4.2.11) follows, whence so does (4.2.10).

Let $\alpha' : X' \rightarrow Q$ denote the canonical morphism and s a regular G -invariant section of L on V . Then $(\overline{\alpha})^{-1}(V)$ is a closed G -invariant subset of $(\alpha')^{-1}(V)$ and by the arguments as in Proposition 3.6.4 or, more precisely Remark 3.6.9, we see that s raised to a suitable power of p extends to a regular G -invariant section of L on $(\alpha')^{-1}(V)$. But since s is already a rational section of L and Q is normal, we see that s can indeed be identified with a G -invariant regular section L on $(\alpha')^{-1}(V)$ i.e. in (4.2.10) we can replace $\overline{\alpha}$ by α' . This establishes (4.2.8), whence the Proposition. \square

Corollary 4.2.12. *Let H be a finite group of Y -automorphisms of Q . Then H lifts to an action of L .*

Proof. The action of H on Q extends to $Q \times_k X^{ss}$ (as well as the line bundle L' on $Q \times_k X^{ss}$ which is the pull back of L), by taking the trivial action on X^{ss} . Since H is a group of automorphisms, we see that the action of H on Q lifts to an action on $X' = Q \times_Y X^{ss}$, i.e. X' is as in the proof of the Proposition. This action of H clearly commutes with the action of G . Hence it acts on the sheaf of invariant sections of L on X' . The assertion now follows from the isomorphism in (4.2.10). \square

5. Reduction to the case stable = semi-stable

In this section we revisit and modify certain technically crucial Lemmas and Propositions in [S2], namely Lemma 3.2, Propositions 5.1 and 5.3 of *ibid.* In this section we allow our group G to be a reductive algebraic group (relaxing our requirement that G be semi-simple). Fix a maximal torus T in G , and a Borel subgroup B of G with $B \supset T$. The notations we use are as follows:

- $\Gamma(T)$ will denote the co-root lattice of T , i.e., $\Gamma(T)$ will denote the abelian group of *one parameter subgroups* (1-PS) $\lambda : \mathbb{G}_m \rightarrow T$ of T .
- $E := \Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- $C(B)$ will denote the Weyl chamber in E associated to B , and $\overline{C(B)}$ will denote its closure in E .

Now for any projective algebraic scheme X with a G -action which is linear with respect to an ample bundle L , and for a fixed $x \in X$, the function $\mu^L(x, \lambda)$ is a integral valued function on $\Gamma(T)$ and we extend this function to an \mathbb{R} -valued function on E by setting

$$\mu^L(x, a\lambda) = a\mu(x, \lambda), \quad a \in \mathbb{R}, \quad \lambda \in \Gamma(T).$$

Lemma 5.1.1. *Let X_1, \dots, X_d be projective algebraic schemes on which G acts, such that for each $i = 1, \dots, d$, the action is linear with respect to an ample line*

bundle L_i on X_i . Then there exist a finite number of closed convex cones C_α contained in $\overline{C(B)}$ (resp. E)—independent of $x_i \in X_i$ —such that

- (1) each C_α is the intersection of a finite number of half spaces in E , the half spaces being of the form $\{x \in E \mid \theta(z) \geq 0\}$, θ being a linear form on E , with integral coefficients (with respect to a given basis) and

$$\overline{C(B)}(\text{resp. } E) = \cup_\alpha C_\alpha;$$

- (2) in every C_α , $\mu^{L_i}(x_i, -)$ is linear for fixed $x_i \in X_i$, $i = 1, \dots, d$.

Lemma 5.1.2. *Let $p: Z \rightarrow X$ be a G invariant morphism between projective algebraic G -schemes, the action being linear with respect to the ample line bundle L on X and M on Z . Write $aL + bM$ for the line bundle $p^*(L)^a \otimes M^b$ ($a, b \in \mathbb{Z}$). Then there exists a finite set $S \subset \Gamma(T) \setminus \{0\}$ such that for every line bundle N of the form $N = aL + bM$, where a and b are positive integers, we have*

- (1) $\mu^N(z, \lambda) \geq 0$, $\forall \lambda \in \Gamma(T) \iff \mu^N(z, \lambda) \geq 0$, $\forall \lambda \in S$.
- (2) If $\mu^N(z, \lambda) \geq 0$, $\forall \lambda \in \Gamma(T)$ and $\mu^N(z, \lambda_o) = 0$ for some $\lambda_o \in \Gamma(T) \setminus \{0\}$, then for some $\lambda_o \in S$, we have $\mu^N(z, \lambda_o) = 0$.

Proof. By Lemma 5.1.1, we can subdivide E into a finite number of closed convex cones C_α (each C_α an intersection of closed half-spaces), independent of $x \in X$ or $z \in Z$ such that $\mu^L(x, -)$ and $\mu^M(z, -)$ are linear on each C_α .

Now, for $z \in Z$, and for $N = aL + bM$, with a, b positive integers, we have

$$\mu^N(z, -) = a\mu^L(x, \lambda) + b\mu^M(z, \lambda), \quad \lambda \in \Gamma(T), x = p(z).$$

Let S_α be a finite set of generators (over \mathbb{R}^+) for the cone C_α . We can choose S_α with integral coordinates, i.e., $S_\alpha \subset \Gamma(T)$. Since $\mu^L(x, -)$ and $\mu^M(z, -)$ are linear in each C_α , therefore $\mu^N(z, -)$ is linear on each C_α . It follows that the finite set $S = \cup_\alpha S_\alpha$ satisfies the assertion of the Lemma. \square

Remark 5.1.3. It is worth pointing out that Lemma 5.1.2 also implies that if $\mu^N(z, \lambda) \leq 0$ for some non-trivial 1-PS λ , then $\mu^N(z, \lambda_o) \leq 0$ for some $\lambda_o \in S$. Indeed, if $\mu^N(z, \lambda_o) > 0$ for every $\lambda_o \in S$, then by part (1), we have $\mu^N(z, \lambda_o) \geq 0$ for every non-trivial 1-PS λ_o . If further, $\mu^N(z, \lambda_o) = 0$ for any non-trivial 1-PS λ_o then by part (1), $\mu^N(z, \lambda_o) = 0$ for some $\lambda_o \in S$, giving the required contradiction.

Proposition 5.1.4. *Let $p: Z \rightarrow X$ be a G -morphism between projective algebraic schemes, the action being linear with respect to an ample bundle L on X and a relatively ample bundle (with respect to $p: Z \rightarrow X$) M on Z . Write $N(a, b) = aL + bM$ for positive integers a, b . Then for $\frac{b}{a}$ sufficiently small, we have*

$$p^{-1}(X^s(L)) \subset Z^s(N(a, b)) \subset Z^{ss}(N(a, b)) \subset p^{-1}(X^{ss}(L))$$

(the second inclusion is obvious).

Proof. First, we can find a_1 and b_1 such that $M_1 = a_1L + b_1M$ is ample on Z . Replacing M_1 by M , we can suppose without loss of generality that M is in fact ample on Z and not merely relatively ample. We are thus in the situation of Lemma 5.1.2. Let $S \subset \Gamma(T) \setminus \{0\}$ be the finite set satisfying the conclusions of *loc.cit.* We have, for $z \in Z$

$$\mu^{N(a,b)}(z, \lambda) = a\mu^L(x, \lambda) + b\mu^M(z, \lambda), x = p(z).$$

Choose positive integers a, b such that

$$\left| \mu^M(z, \lambda) \frac{b}{a} \right| < 1 \text{ for all } \lambda \in S \text{ and } z \in Z.$$

We can do this for the functions $\mu^M(_, \lambda)$, $\lambda \in S$ are finite in number, each continuous on the compact set Z . Now (writing $N = N(a, b)$)

$$\mu^N(z, \lambda) = a \left[\mu^L(x, \lambda) + \frac{b}{a} \mu^M(z, \lambda) \right].$$

Since a is positive, the sign of $\mu^N(z, \lambda)$ is the same as the sign of $\mu^L(x, \lambda) + \frac{b}{a} \mu^M(z, \lambda)$. Since $\mu^L(x, \lambda)$ is an integer and $|(b/a)\mu^M(z, \lambda)| < 1$ for $\lambda \in S$, it follows that for $\lambda \in S$

- (a) $\mu^N(z, \lambda) \geq 0 \implies \mu^L(z, \lambda) \geq 0;$
- (b) $\mu^N(z, \lambda) \leq 0 \implies \mu^L(z, \lambda) \leq 0.$

We first show that $Z^{ss} \subset p^{-1}(X^{ss}(L))$. Suppose on the contrary, there exists $z \in Z^{ss}(N)$ such that $x = p(z) \notin X^{ss}(L)$. Then there exists $x_1 = x \circ g$, $x_1 \in X$, $g \in G$ such that $\mu^L(x_1, \lambda_o) < 0$ for some non-trivial $\lambda_o \in \Gamma(X)$. By our choice of S (cf. Lemma 5.1.2), we may suppose $\lambda_o \in S$. Let $z_1 = z \circ g$ so that $x_1 = p(z_1)$. Since $z \in Z^{ss}(N)$, therefore $z_1 \in Z^{ss}(N)$, whence $\mu^N(z_1, \lambda_o) \geq 0$. However, $\mu^L(x_1, \lambda_o) < 0$, whence by (a) above, $\mu^N(z_1, \lambda_o) < 0$, giving the required contradiction.

Next we show that $p^{-1}(X^s(L)) \subset Z^s(N)$. Suppose we can find a $z \in p^{-1}(X^s(L))$ such that $z \notin Z^s(N)$. Then $x = p(z) \in X^s(L)$, whence

$$(*) \quad \mu^L(x \circ g, \lambda) > 0 \quad (\forall g \in G \text{ and } \forall \lambda_o \in \Gamma(T) \setminus \{0\}).$$

On the other hand, $z \notin Z^s(N)$, whence there exists a $g \in G$ and a $\lambda \in \Gamma(T) \setminus \{0\}$ such that $\mu^N(z \circ g, \lambda) \leq 0$. By Lemma 5.1.2 (see Remark 5.1.3) we conclude that $\mu^N(z \circ g, \lambda_o) \leq 0$ for some $\lambda_o \in S$. By (b) above, this implies that for this g and this λ_o we have $\mu^L(x \circ g, \lambda_o) \leq 0$, contradicting (*). \square

One can modify the proof of Lemma 5.1.2 in an obvious way to get:

Lemma 5.1.5. *Let G act on the projective schemes X_1, \dots, X_l and Y and suppose the action on X_i ($i = 1, \dots, l$) is linear with respect to an ample line bundle L_i on X_i . Suppose further that $\text{Pic}(Y)$ is generated by ample line bundles M_1, \dots, M_r*

and that the action of G on Y is linear with respect to each M_j , $j = 1, \dots, r$. Let $Z_i := X_i \times Y$, $i = 1, \dots, l$. Note that Z_i is a G -scheme with respect to the diagonal action of G and that this action is linear with respect to every line bundle of the form $aL_i + bM$, $M \in \text{Pic}(Y)$, $a, b \in \mathbb{Z}$. Then there exists a finite subset $S \subset \Gamma(T) \setminus \{0\}$ of 1-PS λ of T —independent of $i = 1, \dots, l$ and of $z_i \in Z_i$ —such that for every line bundle $N_i(a, b)$ of the form

$$N_i(a, b)(= N_i) = aL_i + bM \quad (a, b \in \mathbb{N})$$

with M an ample line bundle on Y , we have

- (1) $\mu^{N_i}(z_i, \lambda) \geq 0$, $\forall \lambda \in \Gamma(T) \iff \mu^{N_i}(z_i, \lambda) \geq 0$, $\forall \lambda \in S$;
- (2) If $\mu^{N_i}(z_i, \lambda) \geq 0$, $\forall \lambda \in \Gamma(T)$ and $\mu^{N_i}(z_i, \lambda_0) = 0$ for some $0 \neq \lambda_0 \in \Gamma(T)$, then $\mu^N(z_i, \lambda_0) = 0$ for some $\lambda_0 \in S$.

Proof. By Lemma 5.1.1, we can subdivide E into a finite number of convex cones C_α (each C_α an intersection of closed half-spaces)—independent of $x_i \in X_i$, $i = 1, \dots, l$ and $y \in Y$ —on which $\mu^{L_i}(x_i, -)$, $i = 1, \dots, l$ and $\mu^{M_j}(y, -)$, $j = 1, \dots, r$ are linear.

Write $z_i = (x_i, y)$, $x_i \in X_i$, $y \in Y$. Now $M = \sum_j b_j M_j$ for $b_j \in \mathbb{Z}$. The proof of the Lemma is almost identical to the proof of Lemma 5.1.2 once one observes that

$$\mu^{N_i}(z_i, \lambda) = a\mu^{L_i}(x_i, \lambda) + b\left(\sum_j b_j \mu^{M_j}(y, \lambda)\right) \quad (\lambda \in \Gamma(T)),$$

which implies that on each C_α , $\mu^{N_i}(z_i, -)$ is linear. \square

Conventions 5.1.6. Let L and M be line bundles on a scheme S and consider the positive integral linear combination $N(a, b) = aL + bM$, i.e. $N(a, b) = L^{\otimes a} \otimes M^{\otimes b}$. For discussions involving notions which are stable under “multiplication” of $N(a, b)$ by a positive integer (e.g., ampleness, semi-ampleness, nefness, bigness of $N(a, b)$) we will often write $N(a, b) = N_{\frac{b}{a}} = L + \frac{b}{a}M$. In particular, for such discussions, if ϵ is a positive rational number, the symbol

$$N_\epsilon = L + \epsilon M$$

is a convenient shorthand. This shorthand can be extended to include pairs of maps $f: S \rightarrow T$ and $g: S \rightarrow U$ with L a line bundle on T and M a line bundle on U , so that $L + \epsilon M$ represents $f^*L + \epsilon g^*M$.

Definition 5.1.7. Let Y be a projective G -scheme as in Lemma 5.1.5. Let (X, L) be a G -pair, $Z = X \times_k Y$, and $p: Z \rightarrow X$ the projection to the first factor.

A line bundle M on Y is said to be *stabilizing for* (X, L) if there exists $\epsilon_0 > 0$ such that

- (a) M is ample.
- (b) For $0 < \epsilon \leq \epsilon_0$, with $N_\epsilon := L + \epsilon M$,

- (i) $p^{-1}(X^s) \subset Z^s(N_\epsilon)$ and $p(Z^{ss}(N_\epsilon)) \subset X^{ss}$;
- (ii) $Z^s(N_\epsilon) = Z^{ss}(N_\epsilon)$;
- (iii) If $0 < \epsilon' \leq \epsilon_0$, then $Z^s(N_\epsilon) = Z^s(N_{\epsilon'})$.

If M is stabilizing for (X, L) and ϵ_0 is a positive rational number upper bound as in (b) above, then we say M is ϵ_0 -stabilizing for (X, L) .

Just as Lemma 5.1.2 is used to prove Proposition 5.1.4, one can use Lemma 5.1.5 to prove the following proposition.

Proposition 5.1.8. *Let $X_1, \dots, X_l, Y; L_1, \dots, L_l; M_1, \dots, M_r$ satisfy the hypotheses of Lemma 5.1.5. As in loc.cit., let $Z_i = X_i \times Y$. Suppose that given a finite set $S \subset \Gamma(T) \setminus \{0\}$, there exists an ample line bundle M on Y such that $\mu^M(y, \lambda) \neq 0$ for every $y \in Y$ and $\lambda \in S$. Then there exists an ample line bundle M on Y which is stabilizing for all the G -pairs (X_i, L_i) , $i = 1, \dots, l$.*

Proof. Once we find an ample M which satisfies (ii) in Definition 5.1.7 for ϵ sufficiently small, then by Proposition 5.1.4, (i) of the definition also follows (for this M) for ϵ sufficiently small. We first prove show that (ii) is satisfied for ϵ sufficiently small.

We are in a situation where Lemma 5.1.5 applies. Let S be a finite set of non-trivial elements of $\Gamma(T)$ satisfying the conclusions of Lemma 5.1.5. Note that S is independent of $i \in \{1, \dots, l\}$. By our hypotheses there exists an ample line bundle M on Y such that $\mu^M(y, \lambda) \neq 0$ for every $y \in Y$ and $s \in S$. Let $N_i = N_i(a, b) := aL_i + bM$, for positive integers a, b . For $z_i \in Z_i$, we have, by Lemma 5.1.5

- (i) $\mu^{N_i}(z_i, \lambda) \geq 0$ for every $\lambda \in \Gamma(T) \iff \mu^{N_i}(z_i, \lambda) \geq 0$ for every $\lambda \in S$.
- (ii) If $\mu^{N_i}(z_i, \lambda) \geq 0$ for every $\lambda \in \Gamma(T)$ and $\mu^{N_i}(z_i, \lambda_o) = 0$ for some $\lambda_o \in \Gamma(T) \setminus \{0\}$, then $\mu^{N_i}(z_i, \lambda_o) = 0$ for some $\lambda_o \in S$.

Choose positive integers a, b such that

$$(*) \quad \left| \frac{b}{a} \mu^M(y, \lambda) \right| < 1 \quad \forall \lambda \in S, \quad y \in Y.$$

This is possible because for fixed $\lambda \in S$, $\mu^M(-, \lambda)$ is a continuous function on the compact space Y , and S is finite. Suppose, by way of contradiction, $z_i \in Z_i^{ss}(N_i)$ and $z_i \notin Z_i^s(N_i)$. Then there exists $z'_i \in Z_i^{ss}(N_i)$, $z'_i = z_i \circ g$ for some $g \in G$, and non-trivial $\lambda \in \Gamma(T)$ such that $\mu^{N_i}(z'_i, \lambda) = 0$. By (ii) above, we may assume that $\lambda_o \in S$. Now

$$\mu^{N_i}(z'_i, \lambda_o) = a \left[\mu^{L_i}(z'_i, \lambda_o) + \frac{b}{a} \mu^M(y'_i, \lambda_o) \right] \quad y'_i = p_i(z'_i)$$

and hence, $\mu^{N_i}(z'_i, \lambda_o) = 0$ implies that $\mu^{L_i}(z'_i, \lambda_o) + (b/a) \mu^M(y'_i, \lambda_o) = 0$. By (*) this means that $\mu^{L_i}(z'_i, \lambda_o) = 0$ and $\mu^M(y'_i, \lambda_o) = 0$. But M has been chosen so that $\mu^M(y, \lambda) \neq 0$ for any $y \in Y$ and $\lambda \in S$. This gives the required contradiction.

We have therefore shown that the ample bundle M s=chosen fulfills (i) and (ii) in Definition 5.1.7 .

It remains to that this choice of M satisfies (iii). Suppose μ_1 and μ_2 are integers, and $C = \{\epsilon > 0 : \epsilon|\mu_2| < 1\}$. We claim that if $\mu_1 + \epsilon\mu_2 > 0$ for any $\epsilon \in C$, then $\mu_1 + \epsilon'\mu_2 > 0$ for every element $\epsilon' \in C$. Indeed, let $\epsilon \in C$ be such that $\mu_1 + \epsilon\mu_2 > 0$. Clearly μ_1 cannot be negative by definition of C . If $\mu_1 = 0$, then, clearly $\mu_2 > 0$ whence $\mu_1 + \epsilon'\mu_2 = \epsilon'\mu_2 > 0$ for every $\epsilon' \in C$ (in fact for every $\epsilon' > 0$). If, $\mu_1 > 0$, then $\mu_1 \geq 1$. In this case, since $\mu_1 + \epsilon'|\mu_2| < 1$ for $\epsilon' \in C$, it follows that $\mu_1 + \epsilon'\mu_2 > 0$ for such ϵ' . The same argument shows that $\mu_1 + \epsilon\mu_2 < 0$ for some $\epsilon \in C$ is equivalent to $\mu_1 + \epsilon'\mu_2 < 0$ for all $\epsilon' \in C$.

We will suppress the index $i \in \{1, \dots, l\}$ and, for example, write X for X_i , Z for Z_i , etc. Let $z = (x, y) \in Z = X \times_k Y$. Suppose $z \notin Z^s(N(a, b)) = Z^{ss}(N(a, b))$. Then for some $g \in G$ and $\lambda \in \Gamma(T) \setminus \{0\}$, we have $\mu^{aL+bM}(z \circ g \lambda) < 0$, and by our choice of M , we can, and will, take this λ to lie in S . Let $z' = z \circ g$, and $x' = x \circ g$, $y' = y \circ g$. Then setting $\mu_1 = \mu^L(x', \lambda)$ and $\mu_2 = \mu^M(y', \lambda)$, the above argument gives $\mu^{a'L+b'M}(z', \lambda) < 0$ for any pair of positive integers a' , b' with $b'/a' < \epsilon_0$. In other words, $z' \notin Z^s(N(a', b')) = Z^{ss}(N(a', b'))$. This proves that M is stabilizing for all the (X_i, L_i) . \square

Remark 5.1.9. In [S2, p. 550, Thm. 7.1] it is proven (without assuming geometric reductivity of G) that if (X, L) is a pair on which G acts linearly, with L an ample line bundle on X , and X a normal variety, satisfying $X^s(L) = X^{ss}(L)$, then the geometric quotient of $X^s(L)$ with respect to G exists as a normal projective variety Y and L descends to an ample line bundle on Y . In fact $Y = \text{Proj}(R^G)$, where $R = \bigoplus_{n \geq 0} \Gamma(X, L^n)$. In view of this, if the $Z_i^s(N_i(a, b))$ are non-empty, and \tilde{Z}_i the normalisation of Z_i , we get by part (2) of the Proposition the existence of $\tilde{Z}_i^s(N_i(a, b))//G$ as a geometric quotient for each i . Part (3) of the Proposition shows that the quotients $W_i = W_{i,M} := \tilde{Z}_i^s(N_i(a, b))//G$ for b/a sufficiently small, do not depend on (a, b) , but only on M (and i).

Remark 5.1.10. The hypotheses on M in Proposition 5.1.5 and Proposition 5.1.8 may well be unnecessary. To begin with, note that $\mu^M(y, \lambda)$ makes sense for any $M \in \text{Pic}(Y)$, whether ample or not, provided the action of G on Y lifts to M . Next note that if M and M' are algebraically equivalent line bundles then $\mu^M(y, \lambda) = \mu^{M'}(y, \lambda)$ for every $y \in Y$ and $\lambda \in \Gamma(T)$. Indeed, if M is algebraically equivalent to zero, then by reducing to the case of curves, it is easy to see that $\mu^M = 0$. Finally note that the Neron-Severi group $\text{NS}(Y)$ is finitely generated.

5.2. Applications. Let

$$\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$$

be a quotient data. We fix a Borel subgroup $B \subset G$ and a maximal torus T of G such that $T \subset B$. As in the beginning of Section 5, we denote $\Gamma(T)$ the group of 1-PS of T , and $C(B)$ the positive Weyl chamber associated to (B, T) . Recall that $B \backslash G$ has the following properties [S2, pp. 533–534, Prop. 5.3]:

- $\text{Pic}(B \backslash G)$ is finitely generated by ample line bundles which are linear with respect to the natural action of G on $B \backslash G$.
- Given a finite set $S \subset \Gamma(T) \setminus \{0\}$, there exists an ample line bundle M on $B \backslash G$, linear with respect to G , such that $\mu^M(y, \lambda) \neq 0$ for every $y \in B \backslash G$ and $\lambda \in S$.

We are therefore in a position to apply Proposition 5.1.8 to $Z := X \times_k B \backslash G$ and deduce the existence of an stabilizing line bundle (see Definition 5.1.7) $M \in \text{Pic}(B \backslash G)$ for (X, L) .

Proposition 5.2.1. *Suppose Δ above is a strong quotient data. Then we can find an $\epsilon_0 > 0$ and an ϵ_0 -stabilizing M for (X, L) ($M \in \text{Pic}(B \backslash G)$) such that for $0 < \epsilon \leq \epsilon_0$ and with $N_\epsilon := L + \epsilon M$ the following hold:*

- (i) $Z^s(N_\epsilon) \neq \emptyset$.
- (ii) $Z^s(N_\epsilon) // G$ exists as a geometric quotient and the natural map

$$q: Z^s(N_\epsilon) // G \rightarrow Y$$

is surjective.

- (iii) Let $p: Z \rightarrow X$ be the natural projection. Then, given $x \in X^{ss}$, there exists $x' \in p(Z^s(N_\epsilon))$ such that x' is semi-stably equivalent to x .

Proof. Write $\mathcal{X} = \mathbb{P}(V)$, $\mathcal{Y} = (\mathbb{P}(V)^{ss} // G)_{\text{top}}$, $\mathcal{Z} = \mathcal{X} \times_k B \backslash G$. Let $\tilde{\alpha}: \mathcal{X} \rightarrow \mathcal{Y}$ be the quotient map for semi-stable equivalence. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(V)}(1)$.

By Proposition 5.1.8 we can find an ϵ_0 -stabilizing bundle $M \in \text{Pic}(B \backslash G)$ for $(\mathcal{X}, \mathcal{L})$ for some $\epsilon_0 > 0$. Fix a positive rational number ϵ which is less than or equal to ϵ_0 . To lighten notation we write \mathcal{Z}^s for $\mathcal{Z}^s(N_\epsilon)$ and Z^s for $Z^s(N_\epsilon)$ where N_ϵ is the line bundle $N_\epsilon := \mathcal{L} + \epsilon M$ on \mathcal{Z} . Let $\tilde{p}: \mathcal{Z} \times_k B \backslash G \rightarrow \mathcal{Z}$ be the projection. Since $\tilde{p}^{-1}(\mathcal{Z}^s) \subset \mathcal{Z}^s$ and $\mathcal{Z}^s \neq \emptyset$, it follows that $\mathcal{Z}^s \neq \emptyset$. Now \mathcal{Z} is normal and $\mathcal{Z}^s = \mathcal{Z}^{ss}$ whence by Remark 5.1.9 we have a geometric quotient $\mathcal{W} = \mathcal{Z}^s // G$ which is projective and on to which N_ϵ descends as an ample line bundle. Let $\tilde{\beta}: \mathcal{Z}^s \rightarrow \mathcal{W}$ be the quotient map. There is a natural continuous map $\tilde{q}: \mathcal{W} \rightarrow \mathcal{Y}$ such that the diagram

$$\begin{array}{ccc} \mathcal{Z}^s & \xrightarrow{\tilde{p}} & \mathcal{Z}^{ss} \\ \tilde{\beta} \downarrow & & \downarrow \tilde{\alpha} \\ \mathcal{W} & \xrightarrow{\tilde{q}} & \mathcal{Y} \end{array}$$

commutes. According Lemma 4.1.1 there is a non-empty open subset $\mathcal{U} \subset \mathcal{Y}$ which has a scheme structure such that

$$\tilde{\alpha}^{-1}(\mathcal{U}) \xrightarrow{\text{via } \tilde{\alpha}} \mathcal{U}$$

is a map of schemes, and the structure sheaf on \mathcal{U} is the sheaf of G -invariants of $\mathcal{O}_{\tilde{\alpha}^{-1}(\mathcal{U})}$. Since \mathcal{Y} is irreducible, $\mathcal{U} \cap \tilde{\alpha}(\mathcal{X}^s)$ is a non-empty open set and clearly $\tilde{q}(\mathcal{W}) \supset \mathcal{U} \cap \tilde{\alpha}(\mathcal{X}^s)$. This immediately implies that $\mathcal{W} \rightarrow \mathcal{Y}$ is surjective, for $\mathcal{X}^{ss} \bmod G$ is proper by Corollary 3.6.8. Now Y is a non-empty closed subset of \mathcal{Y} . As a topological space, $Z^s = \mathcal{Z}^s \cap Z = \mathcal{Z}^s \cap \tilde{p}^{-1}(X) = \mathcal{Z}^s \cap \tilde{p}^{-1}(X^{ss})$. The last equality uses the fact that $\tilde{p}(Z^s) = \tilde{p}(Z^{ss}) \subset X^{ss}$. Since X^{ss} is saturated, $X^{ss} = \tilde{\alpha}^{-1}(Y)$, whence, $Z^s = \mathcal{Z}^s \cap \tilde{p}^{-1}(Y) = \tilde{\beta}^{-1}(\tilde{q}^{-1}(Y))$. Since $\tilde{\beta}$ and \tilde{q} are both surjective, $Z^s \neq \emptyset$. This proves (i).

Since $\tilde{\beta}$ maps distinct G -orbits into distinct points of \mathcal{W} , the “orbit space” of Z^s can be identified, as a topological space, can be identified with $\tilde{q}^{-1}(Y)$. It is not difficult to show that this implies the existence of a geometric quotient $\beta: Z^s \rightarrow W$ such that the natural map $W \rightarrow \tilde{q}^{-1}(Y)$ is bijective (as a map of sets). This gives the required surjectivity of $q: W \rightarrow Y$ proving (ii).

Part (iii) follows from (ii). In greater detail first pick $z \in \mathcal{Z}^s$ such that $\tilde{q}(\tilde{\beta})(z) = \tilde{\alpha}(x)$ using the surjectivity of $\tilde{q} \circ \tilde{\beta}$. Then z actually lies in Z^s since $\tilde{\alpha}(x) \in Y$. Now set $x' = \tilde{p}(z) = p(z)$. \square

5.3. Elimination of finite isotropies. In this subsection we summarize the results in [S2, pp. 536–544, § 6]. Let (X, L) be a G -pair with X *normal* and $X^s \neq \emptyset$. According to [ibid., Theorem 6.1 and Remark 6.2] we can find a *normal* G -variety Z and a *finite surjective* G -morphism $p: Z \rightarrow X$ such that (with $X^s = X^s(L)$, and $Z^s = Z^s(p^*(L))$):

- (i) G operates *freely* on Z^s and the geometric quotient $W = Z^s/G$ exists as a normal variety, and the quotient map $\beta: Z^s \rightarrow W$ is a principal bundle, locally trivial in the *Zariski topology*.
- (ii) If $k(X)$ and $k(Z)$ denote the function fields of X and Z respectively, the extension $k(X) \rightarrow k(Z)$ is finite and normal.
- (iv) If Γ is the group of $k(X)$ -automorphisms of $k(Z)$, the canonical action of Γ on Z^s commutes with that of G .
- (v) If W is quasi-projective, $X^s//G$ exists as a quasi-projective variety. (See [ibid., p. 543, Remark 6.1].)

6. Big line bundles

In this section we will show that there is a normal projective variety Q mapping to the stack $[X^{ss}/G]$ which is generically finite and dominant over $Y = (X^{ss}/G)_{\text{top}}$

and such that the pull-back of the line bundle L/G to Q is big. Here L/G is the line bundle on $[X^{ss}/G]$ as in Definition 4.2.5.

6.1. Basic lemmas on bigness and nefness. The following two lemmas are the basic tools for proving bigness and nefness of bundles.

Lemma 6.1.1. *Suppose L and M are line bundles on an algebraic scheme W such that $N = N_\epsilon = L + \epsilon M$ is ample for sufficiently small positive. Then L is nef.*

Proof. Let $C \hookrightarrow W$ be a closed irreducible and reduced curve. Since N_ϵ is ample (for sufficiently small $\epsilon > 0$) $\deg N_\epsilon|_C > 0$. This means that for $\epsilon = \frac{b}{a}$ small, $a \deg L|_C + b \deg M|_C > 0$, i.e.,

$$\deg L|_C > -\epsilon \deg M|_C.$$

Letting ϵ approach zero, we conclude that $\deg L|_C \geq 0$. □

Here is a criterion for bigness of a line bundle on a variety in terms of the nefness of associated bundles on a blow-up of the ambient variety.

Lemma 6.1.2. *Let $\psi: W' \rightarrow W$ be the blow up of an irreducible projective variety W by a coherent ideal sheaf I of \mathcal{O}_W whose support is a finite number of points on W , and suppose $r := \dim W \geq 2$. Let $J = I\mathcal{O}_{W'}$ be the ideal sheaf of the exceptional divisor of the blow-up and L a nef line bundle on W such that $\psi^*(L) + \epsilon J$ is nef on W' for sufficiently small ϵ . Then $L^{(r)} > 0$, i.e., L is big on W .*

Proof. Since L is nef on W , $L^{(r)} \geq 0$. It suffices to show that $L^{(r)} = 0$ leads to a contradiction. The restriction of L to the finite number of points which are blown up is trivial, whence using the “projection formula” (see [K2, p. 296, Proposition 2.11]) we conclude that

$$\psi^*(L)^k \cdot J^{(l)} = 0 \quad (k + l = r, k \geq 1).$$

It follows that

$$(6.1.3) \quad (\psi^*(L) + \epsilon J)^{(r)} = \psi^*(L)^{(r)} + \epsilon^r J^{(r)}.$$

Since $L^{(r)} = 0$, one sees that $\psi^*(L)^{(r)} = 0$. The LHS of (6.1.3) is non-negative since $\psi^*(L) + \epsilon J$ is nef. Hence to get a contradiction it suffices to show that $J^{(r)} < 0$. Let E be the exceptional divisor of the blow-up ψ . One knows that $J|_E$ is ample on E . By the Asymptotic Riemann-Roch (see [K2, p. 208, Corollary 2.14]) we know that if M is a line bundle on a projective variety X and F a coherent sheaf on X with $\dim \text{supp } F = l$, then

$$(6.1.4) \quad \chi(X, M^n \otimes F) = \frac{(M^{(l)} \cdot F)}{l!} n^l + O(n^{l-1}).$$

One needs to set $X = \text{supp } F$ in *loc.cit.* to get the above formula, and this can clearly be done without loss of generality. The LHS of (6.1.4) is a polynomial function in n and l . Next consider the exact sequence

$$0 \rightarrow J \rightarrow \mathcal{O}_{W'} \rightarrow \mathcal{O}_E \rightarrow 0.$$

We have (for $n \gg 0$)

$$\chi(W', J^{n+1}) = \frac{J^{(r)}}{r!} (n+1)^r + O((n+1)^{r-1}),$$

and

$$\chi(W', J^n) = \frac{J^{(r)}}{r!} n^r + O(n^{r-1}).$$

Taking the difference we get

$$\begin{aligned} \chi(W', J^{n+1}) - \chi(W', J^n) &= \frac{J^{(r)}}{r!} [(n+1)^r - n^r] + \cdots + O(n^{r-2}) \\ (6.1.5) \quad &= \frac{J^{(r)}}{r!} r \cdot n^{(r-1)} + O(n^{r-2}) \\ &= \frac{J^{(r)}}{(r-1)!} \cdot n^{(r-1)} + O(n^{r-2}). \end{aligned}$$

On the other hand, we have the exact sequence

$$0 \rightarrow J^{n+1} \rightarrow J^n \rightarrow J^n|_E \rightarrow 0$$

whence

$$-\chi(W', J^{n+1}) + \chi(W', J^n) = \chi(E, J^n|_E).$$

Since $J|_E$ is ample,

$$\chi(E, J^n|_E) = \frac{b}{(r-1)!} n^{r-1} + O(n^{r-2})$$

with $b > 0$. Comparing this with the asymptotic formula in (6.1.5) above we get $b = -J^{(r)}$. In other words $J^{(r)} < 0$, giving the sought for contradiction. \square

6.2. Equivariant Blow-ups. In this sub-section we fix an *irreducible standard quotient data* (see Definition 2.1.3)

$$(6.2.1) \quad \Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$$

For the rest of this section we deal with the following situation: Let u_0 be a point in Y , and consider the reduced closed subscheme C of X which is the closure of $\alpha^{-1}(u_0)$ in X . Then C is a closed G -invariant subscheme of X . Consider the blow-up

$$\theta: X' \rightarrow X$$

of X along C . If $E \hookrightarrow X'$ is the exceptional divisor of the blow-up, and $I(E)$ the resulting ideal sheaf of E in $\mathcal{O}_{X'}$, then E is G -invariant, and $I(E)$ is a G -invariant

invertible $\mathcal{O}_{X'}$ -module which is relatively ample for the map $\theta: X' \rightarrow X$. We then have

Lemma 6.2.2. *There exists $\epsilon_0 > 0$ such that:*

- (1) $L'_\epsilon = \theta^*(L) + \epsilon I(E)$ is ample on X' for $0 < \epsilon \leq \epsilon_0$.
- (2) $(X')^{ss}(L'_{\epsilon_1}) = X^{ss}(L'_{\epsilon_2})$ for $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$. Let the common variety be denoted X^{ss} .
- (3) $(X')^{ss} \subset \theta^{-1}(X^{ss})$. Equivalently, the inverse image under θ of the unstable locus of (X, L) is contained in the unstable locus of (X', N_ϵ) .
- (4) $(X')^{ss} \setminus E = X^{ss} \setminus C$. In particular $(X')^{ss} \neq \emptyset$.

Proof. Since $I(E)$ is relatively ample, the assertion about the ampleness of L'_ϵ is clear. Parts (1) and (2) follows from Proposition 5.1.8. Part (3) is immediate from [R, p. 352, Theorem 2.3(a)]. \square

It is, at this point, convenient for us to extend the notion of *stabilizing* as well as the notion of ϵ_0 -*stabilizing* defined in Definition 5.1.7 to include schemes Z which are not necessarily the product of X and Y but are close to that. This is primarily because we wish to work with Zariski locally trivial principal G -bundles, and such fibrations will be obtained by replacing $X \times_k (B \backslash G)$ by its normalization and then by eliminating finite isotropies. To that end we make the following definition (and remind the reader about the conventions in 5.1.6).

Definition 6.2.3. Let Y be a projective G -scheme as in Lemma 5.1.5, i.e., $\text{Pic}(Y)$ is generated by a finite number of ample line bundles such that the action of G on Y is linear with respect to all these generators. Let Z be a scheme which admits G -invariant morphisms $p: Z \rightarrow X$ and $\pi: Z \rightarrow Y$. We say that a line bundle M on Y is *stablizing* for (X, L, Z) if the universal map $(p, \pi): Z \rightarrow X \times_k Y$ is *finite and surjective* and (a) and (b) in Definition 5.1.7 are satisfied for this Z and this p . Equivalently M is *stablizing* for (X, L, Z) if the universal map $(p, \pi): Z \rightarrow X \times_k Y$ is finite surjective and M is *stabilizing* for (X, L) . Note that M is *stablizing* for (X, L) if and only if it is so for $(X, L, X \times_k Y)$. We will say M is ϵ_0 -*stabilizing* for (X, L, Z) if $Z \rightarrow X \times_k Y$ is finite surjective and M is ϵ_0 -*stabilizing* for (X, L) .

Let ϵ_0 be as in the conclusion of Lemma 6.2.2. For the rest of this discussion we fix a positive rational number ϵ such that $\epsilon \leq \epsilon_0$, and as in *loc.cit.* use L'_ϵ for $\theta^*L + \epsilon I(E)$.

Lemma 6.2.4. *There exists a positive rational number η_0 , a line bundle M on $B \backslash G$, a normal projective G -variety Z together with G -invariant maps $p: Z \rightarrow X$, $\pi: Z \rightarrow B \backslash G$ such that*

- (i) M is η_0 -*stablizing* for (X, L, Z) and (X', L'_ϵ) . Fix $\eta > 0$ with $\eta \leq \eta_0$ and write $N = N_\eta$ for the line-bundle $L + \eta M := p^*(L) + \pi^*(M)$ on Z .

- (ii) $Z^s \neq \emptyset$, where we write $Z^s = Z^s(N)$.
- (iii) G acts freely on Z^s and we have a (Zariski locally trivial) principal G -bundle

$$\beta: Z^s \rightarrow W = W_M$$

where W is a normal projective variety on to which the line bundle $N|_{Z^s}$ descends as a line ample line bundle N_W .

- (iv) A power of the line bundle $p^*(L)|_{Z^s}$ descends to a line bundle L_W on W and L_W is nef on W .

Proof. Applying Proposition 5.1.8 and Proposition 5.2.1 to $(X_1, X_2; L_1, L_2; Y)$ with $X_1 = X$, $X_2 = X'$, $L_1 = L$, $Y = L'_\epsilon$, and $Y = B \setminus G$, we deduce the existence of $\eta_0 > 0$ and a line bundle M on $B \setminus G$ such that M is η_0 -stabilizing for (X, L) and (X', L'_ϵ) and such that $[X \times_k (B \setminus G)]^s(L + \eta M) \neq \emptyset$ for $0 < \eta < \eta_0$ (Proposition 5.2.1 is needed for the last assertion).

Recall that if G acts on an algebraic scheme S , then it acts on S_{red} as well as the normalization \tilde{S} of S_{red} in a canonical way so that the maps $S_{\text{red}} \rightarrow S$ and $\tilde{S} \rightarrow S_{\text{red}}$ are G -invariant.

The scheme Z is obtained in two steps. First normalize $[X \times_k (B \setminus G)]_{\text{red}}$ and then apply the technique of elimination of finite isotropies described in §§ 5.3. Assertions (i)–(iii) follow easily from the results quoted in §§ 5.3.

The first part of part (iv) is straightforward and well-known. Let L_W be such a line bundle. This means that for a suitable $n \geq 1$, $(p^*L^n)|_{Z^s} = \beta^*L_W$. We say L descends to $\frac{1}{n}L_W$ to describe this situation. The second part is a direct application of Lemma 6.1.1. In fact since $N|_{Z^s} = \beta^*N_W$ and $(p^*L^n)|_{Z^s} = \beta^*L_W$ for a suitable $n \geq 1$, therefore a positive power of $(\pi^*M)|_{Z^s}$ descends to a line bundle M_W on W . A little thought shows that

$$N_W = L_W + a\eta M_W$$

where a is a positive rational number. This a is independent of η and hence Lemma 6.1.1 applies. \square

More can be said. Note that L can be replaced by a positive power of itself without affecting the loci $X^{ss}(L)$, $X^s(L)$, $Z^s(N)$ etc. According to 4.2.4, there is a positive integer n such that L^n is trivial on the fibres of α . We replace L by L^n for the rest of this section and assume without loss of generality that L is trivial on the fibres of α . We therefore have a line bundle L/G on the stack $[X^{ss}/G]$ as in Definition 4.2.5. The map $q: W \rightarrow Y$ in the proof of Lemma 6.2.4 then has the following interpretation. Since $\beta: Z^s \rightarrow W$ is a principal bundle and $p: Z^s \rightarrow X^{ss}$ is G -invariant, by definition, we have a classifying map $f_\beta: W \rightarrow [X^{ss}/G]$. As in

Section 4 we have a map $\gamma: [X^{ss}/G] \rightarrow Y$ and a continuous map $q = \gamma \circ f_\beta: W \rightarrow Y$ (q is a map of stratified spaces) such that the diagram

$$(6.2.5) \quad \begin{array}{ccc} Z^s & \xrightarrow{p} & X^{ss} \\ \beta \downarrow & & \downarrow \\ W & \xrightarrow{f_\beta} & [X^{ss}/G] \\ & \searrow q & \nearrow \gamma \\ & & Y \end{array}$$

commutes. According to Proposition 5.2.1 $q: W \rightarrow Y$ is surjective. Note that the line bundle L_W (to which $p^*L|_{Z^s}$ descends) is given by the formula

$$L_W = f_\beta^* L_{/G}.$$

At this point we draw the reader's attention to Lemma 6.2.2, which deals with the blow up of X along the closure of $\alpha^{-1}(u_0)$ in X , and remind the reader that we have fixed an ϵ in the interval $(0, \epsilon_0]$ where ϵ_0 is as in the conclusion of the Lemma.

Proposition 6.2.6. *Let W be the scheme in part (iii) of Lemma 6.2.4. Let $C_W \subset W$ be the reduced closed subscheme of W given by $C_W = q^{-1}(u_0)$, where $q: W \rightarrow Y$ is the map in Diagram (6.2.5). If $W'' \xrightarrow{\psi} W$ is the blow up of W along C_W and J the (coherent invertible) $\mathcal{O}_{W''}$ -ideal of the exceptional divisor E'' of $W'' \xrightarrow{\psi} W$, then there exists a positive rational number a , independent of $\epsilon < \epsilon_0$, such that $\psi^*(L_W) + a\epsilon J$ is nef on W'' .*

Proof. Recall that the line bundle M on $B \backslash G$ is η_0 -stabilizing for (X', L'_ϵ) (see Lemma 6.2.2 (i)). Set

$$Z' := Z \times_k X'.$$

Then we have a finite surjective G -invariant map $Z' \rightarrow X' \times_k B \backslash G$ and hence M is η_0 stabilizing for (X', L'_ϵ, Z') . We have a commutative diagram

$$(6.2.7) \quad \begin{array}{ccc} Z' & \xrightarrow{p'} & X' \\ \theta' \downarrow & & \downarrow \theta \\ Z & \xrightarrow{p} & X \end{array}$$

Let $\pi': Z' \rightarrow B \backslash G$ be the map $\pi \circ \theta'$. Fix $\eta \in (0, \eta_0]$ and write

$$N' (= N'_\eta = N'_{\eta, \epsilon}) := p'^* L'_\epsilon + \eta \pi'^* M = L'_\epsilon + \eta M.$$

We assume Z' is normal and G acts freely on $(Z')^s(N')$ by first replacing Z' by its normalization and then eliminating finite isotropies (see §§ 5.3). M remains

η_0 -stabilizing for (X', L'_ϵ, Z') through all these modifications. We have a finite surjective map

$$\varphi: Z' \rightarrow Z \times_k X' \quad (\varphi = (\theta', p')).$$

Set

$$\begin{aligned} (Z')^s &= (Z')^s(N') \\ (Z^s)' &= \varphi^{-1}(Z^s \times_{X^{ss}} (X')^{ss}). \end{aligned}$$

(Recall $(X')^{ss} := (X')^{ss}(L'_\epsilon)$. See Lemma 6.2.2.) Note the difference between $(Z^s)'$ and $(Z')^s$ —they need not be the same, though both are G -invariant open subvarieties of Z' . After all these modifications, we still have the commutative diagram (6.2.7). The map $\theta': Z' \rightarrow Z$ is more or less that blow up of Z along $p^{-1}(C)$ where, recall, C is the centre of the blow-up $\theta: X' \rightarrow X$.

Since M is η_0 -stabilizing for (X', L'_ϵ, Z') , we have $(Z')^s(N') = (Z')^{ss}(N')$. Further, Z' is normal and G operates freely on $(Z')^s$. We therefore have (Zariski locally trivial) principal G -bundle

$$\beta': (Z')^s \rightarrow W'$$

with W' normal projective, and such that N' descends to an ample line bundle $N'_{W'}$ on W' . Note that $N'_{W'}$ depends on ϵ and η , and if we wish to draw attention to this, we write $N'_{W', \eta}$, or $N'_{W', \eta, \epsilon}$. Since L is trivial on the fibres of $\alpha: X^{ss} \rightarrow Y$, the line bundle $p'^*\theta^*L$ descends to a line bundle $L_{W'}$ on W' . In fact

$$L_{W'} = f_{\beta'}^* L/G$$

where $f_{\beta'}: W' \rightarrow [X^{ss}/G]$ is the classifying map for the data

$$((Z')^s \xrightarrow{\beta'} W', (Z')^s \rightarrow X^{ss})$$

consisting of a principal bundle and a G -invariant map, the second map being induced by the composite $\theta \circ p'$.

The exceptional divisor E of $\theta: X' \rightarrow X$ pulls back to locally principal (i.e., Cartier) effective divisor $E' = p'^{-1}(E)$ on Z' . This divisor E' can be related to the centre C_W of the blow-up $\psi: W'' \rightarrow W$ in the following way: We have a composite $(Z^s)' \rightarrow Z^s \xrightarrow{\beta} W$. The inverse image of C_W in $(Z^s)'$ under this composite is then $E'|_{(Z^s)'}$. Since the latter is an effective Cartier divisor, by the universal property of blow-ups, we have a unique map

$$\beta'': (Z^s)' \rightarrow W''$$

such that $(\beta'')^{-1}(E'') = E'|_{(Z^s)'}$.

The spaces $(Z^s)'$ and $(Z')^s$ are open subsets of Z' and their intersection \tilde{Z} in Z' is a G -invariant variety. The geometric quotient $\widetilde{W} = \tilde{Z}/G$ exists as a normal quasi-projective variety, being an open subvariety of $W' = (Z')^s/G$. Since the

composite $\tilde{Z} \rightarrow (Z^s)' \xrightarrow{\beta''} W''$ is G -invariant (G acting trivially on W'') and \widetilde{W} is a categorical quotient we have a (dominant) map $\widetilde{W} \rightarrow W''$ such that the composite $\tilde{Z} \rightarrow (Z^s)' \rightarrow W''$ agrees with the composite $\tilde{Z} \rightarrow \widetilde{W} \rightarrow W''$.

Let \overline{W} be the scheme theoretic closure of the locally closed embedding

$$\widetilde{W} \xrightarrow{\text{diag}} W'' \times_k W'.$$

Let $\mu: \overline{W} \rightarrow W''$, $\lambda: \overline{W} \rightarrow W'$, be the projections.

The situation is best described by the diagram below where Y' is the topological space consisting of semi-stable equivalence classes on $(X')^{ss}$ and $\alpha': (X')^{ss} \rightarrow Y'$ the resulting map. The other arrows, not defined earlier, are as described after the diagram:

$$\begin{array}{ccccccc}
 & & Z^s & \xleftarrow{\theta'} & (Z^s)' & & \\
 & \swarrow p & \downarrow & & \swarrow p' & \downarrow p' & \\
 X^{ss} & \xleftarrow{\theta} & (X')^{ss} & \xleftarrow{p'} & (Z')^s & & \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \beta'' & & \\
 & \swarrow q & W & \xleftarrow{\psi} & W'' & \xleftarrow{\mu} & \overline{W} \\
 & & \downarrow \alpha' & & \downarrow \beta' & & \downarrow \lambda \\
 Y & \xleftarrow{\phi} & Y' & \xleftarrow{q'} & W' & &
 \end{array}$$

The map $q': W' \rightarrow Y'$ is the obvious continuous map (the exact analogue of $q: W \rightarrow Y$), $\phi: Y' \rightarrow Y$ the map induced by the equivariant G -map θ .

The bottom rectangle in the diagram commutes for the following reason: First, we have a commutative diagram

$$\begin{array}{ccccc}
 Z^s & \xleftarrow{\theta'} & (Z^s)' & \xrightarrow{\quad} & \tilde{Z} \\
 p \downarrow & & \downarrow p' & & \downarrow \\
 X^{ss} & \xleftarrow{\theta} & (X')^{ss} & \xleftarrow{p'} & (Z')^s
 \end{array}$$

from which it follows that

$$q \circ \psi \circ \mu|_{\widetilde{W}} = \phi \circ q' \circ \lambda|_{\widetilde{W}}.$$

with $\widetilde{W} \subset \overline{W}$ as above (i.e. $\widetilde{W} = (Z^s)' \cap (Z')^s // G$). Since the diagonal map $Y \rightarrow Y \times Y$ is a closed immersion (see Proposition 3.6.3), the locus in \overline{W} where the two maps $q \circ \psi \circ \mu$ and $\phi \circ q' \circ \lambda$ are equal is a closed subset of \overline{W} . It follows, since

\widetilde{W} is dense in \overline{W} , that the bottom rectangle of the 3-dimensional diagram above commutes.

The remaining sub-rectangles of the diagram clearly commute.

Now using $P_1 := \overline{W} \times_W Z^s$ and $P_2 := \overline{W} \times_{W'} (Z')^s$, we have principal G -bundles $P_1 \rightarrow \overline{W}$ and $P_2 \rightarrow \overline{W}$ and P_1 and P_2 have G -invariant maps to X^{ss} , whence we have classifying maps

$$\overline{W} \begin{matrix} \xrightarrow{f_1} \\ \xRightarrow{\quad} \\ \xrightarrow{f_2} \end{matrix} [X^{ss}/G].$$

If $\gamma: [X^{ss}/G] \rightarrow Y$ is the map in (4.2.3) then commutativity of the bottom rectangle of the 3-dimensional diagram amounts to saying that

$$\gamma \circ f_1 = \gamma \circ f_2.$$

By Proposition 4.2.6 it then follows that $f_1^* L_{/G} = f_2^* L_{/G}$. This is the same as saying:

$$\mu^*(\psi^*(L_W)) \cong \lambda^* L_{W'}.$$

Recall that the ideal sheaf of the exceptional divisor E of $\theta: X' \rightarrow X$ was denoted I . Let $I' = p^* I|_{(Z')^s}$. A positive power, say $(I')^l$ descends to an invertible ideal sheaf $I_{W'}$ of $\mathcal{O}_{W'}$. We express this by saying I' descends to $\frac{1}{l} I_{W'}$. Similarly the pull back of M to $(Z')^s$ descends to, say $\frac{1}{m} M_{W'}$. We point out that $L'_W + \frac{\epsilon}{l} I_{W'}$ is nef since $N'_{W', \eta, \epsilon} = L_{W'} + \frac{\epsilon}{l} I_{W'} + \frac{\eta}{m} M_{W'}$ is ample, whence Lemma 6.1.1 applies. This in turn means that $\lambda^*(L_{W'} + \frac{\epsilon}{l} I_{W'})$ is nef.

We have, therefore, two coherent ideal sheaves, $\mu^* J$ and $\lambda^* I_{W'}$, of $\mathcal{O}_{\overline{W}}$. Moreover, the closed topological subspaces of \overline{W} underlying the closed subschemes defined by these two ideals are the same. Since $\mu^* J$ and $\lambda^* I_{W'}$ are invertible ideal sheaves, whence locally principal, and \overline{W} is normal, we have positive integers r and s such that $\mu^* J^r = \lambda^* I_{W'}^s$. Choosing $a = \frac{r}{sl}$ we see that

$$\mu^*(\psi^* L_W + a\epsilon J) = \lambda^*(L_{W'} + \frac{\epsilon}{l} I_{W'}).$$

We have argued that the right side is nef. Hence so is $\psi^* L_W + a\epsilon J$. \square

Corollary 6.2.8. *If C is a closed integral curve in W such that $q|_C$ is not a constant, then $\deg(L_W|_C) > 0$.*

Proof. Without loss of generality we can assume $\dim W > 1$ by embedding W in a higher dimensional variety (by enlarging V for example, i.e., by considering the embedding of X via the complete linear system $\Gamma(X, L^n)$ for $n \gg 0$). Thus C is finite over Y and $\dim W > 1$. Pick $u_0 \in Y$ and consider the blow-up $\psi: W'' \rightarrow W$ of the Proposition. Let $C' \hookrightarrow W''$ be the proper transform of C . By the Proposition we have $\deg(\psi^*(L_W + \epsilon J)|_{C'}) \geq 0$. Since $\deg(J|_{C'}) < 0$, this implies that $\deg(\psi^* L_W|_{C'}) > 0$, whence $\deg(L_W|_C) > 0$. \square

7. Geometric Reductivity

Let $\Delta = (X, L, \mathbb{P}(V), X^{ss} \xrightarrow{\alpha} Y)$ be an irreducible standard quotient data.

All that remains to complete the proof of Theorem 2.3.8 (i.e., to show G is geometrically reductive) is to find a map $f: Q \rightarrow [X^{ss}/G]$ as in (2.2.1) satisfying conditions (1)–(7) in §§2.2. To that end pick a generic quotient U for the data Δ and reconsider Diagram (6.2.5) above, namely:

$$\begin{array}{ccc}
 Z^s & \xrightarrow{p} & X^{ss} \\
 \beta \downarrow & & \downarrow \\
 W & \xrightarrow{f_\beta} & [X^{ss}/G] \\
 & \searrow q & \nearrow \alpha \\
 & & Y
 \end{array}$$

The map $q^{-1}(U) \rightarrow U$ is a morphism of schemes, for the pull-back by p of a G -invariant regular function on X^{ss} descends to a regular function on $q^{-1}(U)$. The map $q^{-1}(U) \rightarrow U$ is proper since W is projective and by Corollary 3.6.8 $X^{ss} \bmod G$ is proper. We can certainly find a closed subvariety of $q^{-1}(U)$ which is generically finite over U . By normalizing the closure of this variety in W and then applying Lemma 4.1.6 we get a normal projective variety Q and a map $f: Q \rightarrow [X^{ss}/G]$ such that resulting map $Q \rightarrow Y$ is generically finite, and satisfies properties (1), (2) and (5) required of the map (2.2.1). We continue to use the symbol q for the map $Q \rightarrow Y$.

Write L_Q for $L_W|_Q = f^*L/G$. By Lemma 6.2.4 (iv) we see that L_Q is nef on Q . Suppose $\dim Q > 1$. Pick a point $u_0 \in U$ such that $q^{-1}(u_0) \subset Q$ is finite. Blow up Q along this inverse image. According to Proposition 6.2.6, if J is the ideal of the exceptional divisor of the blow-up $g: Q' \rightarrow Q$ along this inverse image, then $g^*L_Q + \epsilon J$ is nef for sufficiently small positive values of ϵ , whence by Proposition 6.1.2 L_Q is nef and big on Q . If $\dim Q = 1$, then according to Corollary 6.2.8 $\deg L_Q > 0$, whence L_Q is nef and big (in fact ample). So in every case L_Q is nef and big on Q . This establishes (3) of §§2.2. Property (6) follows from the manner in which Z and Z^s were found. As for property (4), this follows directly from Corollary 6.2.8.

It remains to prove (7) of §§2.2 for our map $f: Q \rightarrow [X^{ss}/G]$. By replacing L by a positive power of itself if necessary, we may assume L_Q descends to a line bundle $L_{\overline{Q}}$ on $\overline{Q} := Q/\Gamma$. We have to show that if $t \in \Gamma(\overline{Q}, L_{\overline{Q}})$, then for some $n \gg 0$, with $n = p^m$, t^n comes from a G -invariant section on the pull-back of L^n on the normalisation \tilde{X}^{ss} of X^{ss} . The proof of this runs along the same lines as that of (4.2.10) in the proof of Proposition 4.2.6. Let $g: \tilde{X}^{ss} \rightarrow X^{ss}$ be the normalisation map and let $\tilde{L} = g^*L$. Let s be the G -invariant meromorphic section of \tilde{L} induced

by t . (We point out that t can be regarded as a (G, Γ) -invariant section of $\tilde{\pi}_* \tilde{\pi}^* \tilde{L}$.) Let D be the polar divisor of s and let $x_1 \in D$ be such that $s(x_1) = \infty$. Let $y = \alpha g(x_1)$. Then as in (4.2.11) in the proof of Proposition 4.2.6 we see that for any $x \in (\alpha \circ g)^{-1}(y)$, $s(x) = \infty$. Now on the open set $(\alpha \circ \pi)^{-1}(U) = (q \circ \beta)^{-1}(U)$, the section $\tilde{\pi}^*(s)$ coincides with the section $\beta^* \varphi^*(t)$, where $\varphi: Q \rightarrow \overline{Q}$ is the natural map. Further since $q: W \rightarrow Y$ is surjective $\tilde{\pi}(P)$ meets $(\alpha \circ g)^{-1}(y)$. Thus we have $x_2 \in \tilde{\pi}(P)$, with $\alpha(g(x_2)) = y$. Since $\beta^* \varphi^*(t)$ is regular, we see that $s(x_2)$ cannot be ∞ . This leads to a contradiction and we are done.

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